Two-port circuits

With the idea of the Thevenin (or Norton) equivalent, we saw that we could represent the behavior of a circuit at a port (pair of nodes) using a simple source-resistor combination.

What if we have two ports? A circuit that has an input and an output would need two ports – for example, an amplifier.

Put a Thevenin at each port.

Not really correct – just two one ports.
Need to use dependent sources. Now port 1 is connected to port 2 (the voltage at port 2 affects port 1) and vice-versa. This is the two-port version of the Thevenin equivalent idea. We can use this to model circuits that have an input and output.

If desired, we can do a source transformation at each end.
Or mix and match...

Use whatever combination is most advantageous in doing the circuit analysis.

The general convention is define the current as inward at each port, even if we know that it is flowing out.
An example
Consider a two port circuit as shown below, with a voltage source attached to the left and a load resistor attached to the right.

\[
Y = \frac{R_L}{R_L + R_2} a_{12} v_1 \\
v_1 = i_2 R_1 + a_{21} v_2 = \left( \frac{V_S - a_{21} v_2}{R_1 + R_S} \right) R_1 + a_{21} v_2 \\
R_L + R_2 \frac{v_2}{a_{12}} = \left( \frac{V_S - a_{21} v_2}{R_1 + R_S} \right) R_1 + a_{21} v_2 \quad \text{Solve for } v_2.
\]

\[
v_2 = \frac{R_1 R_L a_{12}}{(R_1 + R_S) (R_L + R_2) + a_{12} a_{21} [R_1 R_L - R_L (R_1 + R_S)]} V_S = 4.167 \text{ V}
\]
\[ v_2 = \frac{R_1 R_L a_{12}}{(R_1 + R_S)(R_L + R_2) + a_{12} a_{21} [R_1 R_L - R_L (R_1 + R_S)]} V_S \]

If \( a_{21} = 0 \)

\[ v_2 = \frac{R_1 R_L a_{12}}{(R_1 + R_S)(R_L + R_2)} V_S = 4.545 \text{ V} \]

\[ v_1 = \frac{R_1}{R_1 + R_S} V_S \]

\[ v_2 = \frac{R_L}{R_L + R_2} a_{12} v_1 \]

\[ v_2 = \left( \frac{R_S}{R_1 + R_S} \right) a_{12} \left( \frac{R_L}{R_2 + R_L} \right) V_S \]
For amplifiers, the simpler model with $a_{21} = 0$ is generally sufficient to provide a good description of the circuit’s behavior.

In general, we need to include all four components if we want a complete description of an unknown circuit.
Determining the parameters for a two-port

Similar to finding the parameters for a one-port Thevenin. Since there are four parameters, we expect four measurements or calculations.

But first, note that since there is no “internal” independent source, we need to provide an independent source at one side so that we can get a response at the other side. We can call this a “test source”. It can be either type (voltage or current) and it can have any value that we like. Typically, we might choose 1 V or 1 A, or we might leave it in symbolic form. It doesn’t really matter because the response will always be proportional to the value of the test source.

Like the Thevenin approach, we might expect to look for open-circuit voltage and short-circuit currents at each side to determine the voltages and resistances.
However, it turns out that the open-circuit measurements are not really necessary.

1. Apply a test voltage, $V_t'$ at port 1. $v_1 = V_t'$.
2. Short the terminals at port 2, making $v_2 = 0$. This has the effect of making the source $a_{21}v_2 = 0$.
3. Measure, or calculate, the currents at the two ports. (Note that we have changed the direction of $i_2$.)
Now reverse the experiment:

4. Apply a test voltage, $V_t''$ at port 2. $v_2 = V_t''$.

5. Short the terminals at port 1, making $v_1 = 0$. This has the effect of making the source $a_{12}v_1 = 0$.

6. Measure, or calculate, the currents at the two ports. (Note that we have changed the direction of $i_1$.)

From the measured (calculated) values, we can determine the four parameters for the two-port.
Example

Find the two-port equivalent for the simple T-network shown below.

1. Apply a test voltage at \( v_1 \). Short circuit \( v_2 \). Calculate the terminal currents.

\[
R_1 = \frac{V_t'}{i_1'} = R_a + R_b \| R_c = 1095\Omega \\
R_{2/12} = \frac{V_t'}{i_2'} = R_c \frac{R_a + R_b \| R_c}{R_b \| R_c} = 1150\Omega
\]
2. Apply a test voltage at $v_2$. Short circuit $v_1$. Calculate the terminal currents.

\[
\begin{align*}
R_2 &= \frac{V''}{i''} = R_c + R_a \parallel R_b = 767 \Omega \\
R_1/a_{21} &= \frac{V''}{i''} = R_a \frac{R_c + R_a \parallel R_b}{R_a \parallel R_b} = 1150 \Omega \\
a_{12} &= \frac{R_2}{R_2/a_{12}} = \frac{767 \Omega}{1150 \Omega} = 0.667 \\
a_{21} &= \frac{R_1}{R_1/a_{21}} = \frac{1095 \Omega}{1150 \Omega} = 0.952
\end{align*}
\]
Example 2

Calculate the two-port parameters for the circuit. Note the symmetry.

1. Apply a test voltage at $v_1$. Short circuit $v_2$. ($R_e$ is shorted out.)
   Calculate the terminal currents.

\[
\begin{align*}
   i_1 &= \frac{V_t'}{R_{eq}} = \frac{V_t'}{R_a \parallel [R_b + R_c \parallel R_d]} \\
   R_1 &= \frac{V_t'}{i_2'} = R_a \parallel [R_b + R_c \parallel R_d] = 50 \Omega \\
   i_2' &= \frac{v_{Rc}}{R_d} = \frac{R_c \parallel R_d}{R_b + R_c \parallel R_d} \frac{V_t'}{R_d} \\
   \frac{R_2}{a_{12}} &= \frac{V_t'}{i_2'} = R_d \frac{R_b + R_c \parallel R_d}{R_c \parallel R_d} = 300 \Omega
\end{align*}
\]
2. Short circuit $v_1$.  Apply a test voltage at $v_2$.  Calculate the terminal currents.

\[ i''_2 = \frac{V'_t}{R_{eq}} = \frac{V''_t}{R_e || [R_d + R_c || R_b]} \]

\[ R_2 = \frac{V''_t}{i''_2} = R_e || [R_d + R_c || R_a] = 50\Omega \]

\[ i''_1 = \frac{v_{RC}}{R_b} = \frac{R_c || R_b}{R_d + R_c || R_b} \frac{V''_t}{R_b} \]

\[ R_1 \frac{a_{21}}{a_{21}} = \frac{V''_t}{i''_1} = R_b \frac{R_d + R_c || R_b}{R_c || R_b} = 300\Omega \]

\[ a_{21} = \frac{R_1}{R_1/a_{21}} = \frac{50\Omega}{300\Omega} = 0.167 \]

\[ a_{12} = \frac{R_2}{R_2/a_{12}} = \frac{50\Omega}{300\Omega} = 0.167 \]

As expected, also symmetric.
Example 3

Calculate the two-port parameters for the circuit. Note that the dependent source will probably make this circuit asymmetric.

\[ \gamma = 0.5 \text{ S} \]

1. Short circuit \( v_2 \). (\( R_d \) is shorted out.) Apply a test voltage at \( v_1 \). Calculate the terminal currents.

\[ i'_1 = \frac{V'_t - v_x}{R_a} = \frac{V'_t}{10 \Omega} \]

\[ i'_2 = i_{Rc} + \gamma v_{Rb} \]

\[ = \frac{v_x}{R_c} + \gamma v_x \]

\[ = \frac{V'_t}{2R_c} + \frac{\gamma}{2} V'_t = \frac{V'_t}{3.333 \Omega} \]
2. Short circuit $v_1$. Apply a test voltage at $v_2$. Calculate the terminal currents.

\[
\begin{align*}
    i_1'' &= \frac{v_x}{R_a} = \frac{V_t''}{20\Omega} \\
    i_2'' &= \frac{v_x}{R_d} + \frac{\gamma v_{Rb}}{R_c} \\
    i_2'' &= \frac{V_t''}{R_d} + \frac{V_t'' - v_x}{R_c} \\
    i_2'' &= \left[ \frac{1}{R_d} + \frac{3}{4R_c} - \frac{\gamma}{4} \right] V_t'' = \frac{V_t''}{60\Omega} \\
    R_1 &= \frac{V_t'}{i_1''} = 10\Omega \\
    R_2 &= \frac{V_t''}{i_2''} = 60\Omega \\
    a_{12} &= \frac{R_2}{V_t'/i_2''} = 18 \\
    a_{21} &= \frac{R_1}{V_t''/i_1''} = 0.5
\end{align*}
\]
Example 4

Calculate the two-port parameters for the circuit.

1. Apply a test voltage at $v_1$. Short circuit $v_2$. Calculate the terminal currents.

\[
\begin{align*}
V'_t &= R_i || R_f = R_1 \\
\frac{V'_t}{i'_1} &= \frac{R_i}{R_i || R_f} = \frac{R_i}{R_1} \\
\frac{V'_t}{i'_2} &= \left( \frac{R_o}{A} \right) \left( \frac{R_0}{R_f} \right) = \frac{R_2}{a_{12}}
\end{align*}
\]
2. Apply a test voltage at $v_2$. Short circuit $v_1$. Calculate the terminal currents.

\[ i_1'' = \frac{V_{t}''}{R_f} \]

\[ \frac{V_{t}''}{i_2''} = R_o || R_f = R_2 \]

\[ R_i \text{ shorted, } v_a = 0. \]

\[ R_1 = \frac{V_{t}'}{i_1''} = R_i || R_f = 8.33 \, k\Omega \]

\[ R_2 = \frac{V_{t}''}{i_2''} = R_o || R_f = 99.8 \, \Omega \]

\[ a_{21} = \frac{R_2}{V_{t}' / i_2''} = \frac{R_o || R_f}{(\frac{R_o}{A}) || R_f} = 99.0 \]

\[ a_{21} = \frac{R_1}{V_{t}'' / i_1''} = \frac{R_i || R_f}{R_f} = 0.833 \]
To summarize:

1. Apply a test voltage, $V_t'$, at port 1 so that $v_1 = V_t'$.

2. Short the terminals at port 2, making $v_2 = 0$. This has the effect of making the source $a_{21}v_2 = 0$.

3. Measure, (or calculate), the currents at the two ports. The two measurements (calculations) give $R_1$ directly and the ratio $\frac{R_2}{a_{12}}$.

4. Apply a test voltage, $V_t''$ at port 2 so that $v_2 = V_t''$.

5. Short the terminals at port 1, making $v_1 = 0$. This has the effect of making the source $a_{12}v_1 = 0$.

6. Measure, or calculate, the currents at the two ports. The two measurements (calculations) give $R_2$ directly and the ratio $\frac{R_1}{a_{21}}$.

7. Calculate $a_{12}$ and $a_{21}$ from the acquired data.

Note: If a circuit is symmetric, it’s two-port equivalent will also be symmetric.

Note: If there are no dependent sources, only resistors, the circuit is passive and $a_{12} < 1$ and $a_{21} < 1$. If there are dependent sources, then circuit may be active, such that either $a_{12} > 1$ or $a_{21} > 1$ or both.
The form of two-port equivalent that we have used to this point is just one of several different forms. There are other ways to view the relationship between the terminal currents and voltages. What we have done is to write the currents in terms of the voltages.

\[
\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

\[
y_{11} = \left. \frac{i_1}{v_1} \right|_{v_2=0} \quad y_{12} = \left. \frac{i_1}{v_2} \right|_{v_1=0}
\]

\[
y_{21} = \left. \frac{i_2}{v_1} \right|_{v_2=0} \quad y_{22} = \left. \frac{i_2}{v_2} \right|_{v_1=0}
\]
Express $v$ in terms of $i$ (impedance parameters)

\[
v_1 = z_{11}i_1 + z_{12}i_2 \\
v_2 = z_{21}i_1 + z_{22}i_2
\]

\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
\]

Units of $\Omega$ for all parameters.

Express output in terms of input (transmission parameters)

\[
v_2 = t_{11}v_1 + t_{12}i_1 \\
i_2 = t_{21}v_1 + t_{22}i_1
\]

\[
\begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix}
\]

Different units for different parameters.

Express input in terms of output (reflection parameters)

\[
v_1 = r_{11}v_2 + r_{12}i_2 \\
i_1 = r_{21}v_2 + r_{22}i_2
\]

\[
\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}
\]

Different units for different parameters.
Express $v_1$ and $i_2$ terms of $v_2$ and $i_1$ (hybrid parameters)

$$v_1 = h_{11} v_2 + h_{12} i_1$$
$$i_2 = h_{21} v_2 + h_{22} i_1$$

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} v_2 \\ i_1 \end{bmatrix}$$

Different units for different parameters.

Express $v_2$ and $i_1$ terms of $v_1$ and $i_2$ (also known as hybrid parameters)

$$v_2 = g_{11} v_1 + g_{12} i_2$$
$$i_1 = g_{21} v_1 + g_{22} i_2$$

$$\begin{bmatrix} v_2 \\ i_1 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

Different units for different parameters.
With these different forms, the $i$-$v$ relationships become more abstract. We do not even need to know about the specific components making up the equivalent two-port network. The matrix elements tell us everything we need.

If we know set of matrix parameters, we can convert to any other set. See the giant conversion tables in the text book. (For math nerds in the class: derive the conversion formulas. For example, convert $y$-parameters to $t$-parameters. It’s not hard, just tedious linear algebra.)

For the most part, we will only use $y$-parameters to express the typical equivalent circuit for amplifiers. With that starting point, we can shift to other forms using source transformations, if needed.