Thevenin equivalent circuits

We have seen the idea of equivalency used in several instances already.

$V_{S1}$

$V_{S2}$

$same as$

$same as$

$same as$

$same as$

$same as$

$same as$

$same as$

$same as$

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$same as$
The behavior of any circuit, with respect to a pair of terminals (port) can be represented with a Thevenin equivalent, which consists of a voltage source in series with a resistor.

Need to determine $V_{Th}$ and $R_{Th}$ so that the model behaves just like the original.
Ideas developed independently (Thevenin in 1880’s and Norton in 1920’s). But we recognize the two forms as identical because they are source transformations of each other. In EE 201, we won’t make a distinction between the methods for finding Thevenin and Norton. Find one and we have the other.
Attach various load resistors to the original circuit. Do the same for the equivalent circuit. For each load resistance, calculate the load voltage (and current and power) for each of the circuits. The results are identical. In terms of the load that is attached at the port, the two circuits are indistinguishable.

Check it yourself.
Determining the Thevenin (or Norton) components

How to find $V_{Th}$ and $R_{Th}$?

Need two components, so two measurements or calculations should suffice. Use two different load resistors.

2 equations, 2 unknowns:

\[
V_{Th} = \frac{v_1 v_2 (R_1 - R_2)}{R_1 v_2 - R_2 v_1}
\]

\[
R_{Th} = \frac{R_1 R_2 (v_1 - v_2)}{R_1 v_2 - R_2 v_1}
\]
More directly: open-circuit voltage, short-circuit current

1. Leave port open-circuited. \((R_L \rightarrow \infty, i_L = 0)\) Measure open-circuit voltage.

\[ v_{oc} = V_{Th} \]

open-circuit voltage is a direct measure of \(V_{Th}\).

2. Short the output port. \((R_L = 0, v_L = 0)\) Measure short-circuit current.

\[ i_{sc} = \frac{V_{Th}}{R_{Th}} \]

\[ R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{v_{oc}}{i_{sc}} \]

Note, that \(i_{sc}\) can also be interpreted as a direct measurement of \(I_N\): \(i_{sc} = I_N\).
Calculating Thevenin equivalent

The open-circuit voltage / short-circuit current approach can be used to calculate the Thevenin equivalent for a known circuit.

Consider the circuit from slide 4:

Open-circuit voltage – Use whatever method you prefer. We’ll use node voltage in this case.

\[
V_{Th} = V_{oc} = 12 \text{ V}. 
\]
Short-circuit current – Use whatever method you prefer. We’ll use node voltage in this case. But proceed carefully – the short circuit introduces some unusual wrinkles into the circuit analysis.

a: Because of the short circuit, \( v_a = 0 \! \)!

b: Because of the short, \( v_{R2} = 0 \) and \( i_{R2} = 0 \). So \( R_2 \) plays no role and can be removed.

\[
i_{sc} = i_{R1} + I_S = \frac{V_S - v_a}{R_1} + I_S
\]

\[
= \frac{V_S}{1.5k\Omega} + 6mA = 12mA
\]
Alternate method for $R_{Th}$
If the circuit consists of independent sources and resistors only, then the Thevenin resistance can also be found by de-activating the independent sources and finding the equivalent resistance as seen from the port.

De-activate the sources

$$R_{Th} = R_1 || R_2 = (1.5 \text{k}\Omega) || (3 \text{k}\Omega) = 1 \text{k}\Omega$$
Summary

To measure $V_{Th}$ and $R_{Th}$

1. Use a voltmeter to measure the open-circuit voltage at the port of the circuit: $v_{oc} = V_{Th}$.

2. Connect a short circuit across the output and use an ammeter to measure the short-circuit current: $i_{sc} = I_{N}$.

3. Calculate $R_{Th} = V_{Th} / I_{N}$.

Note that shorting the output may not always be practical. For example, some devices may have over-current protection circuitry that prevents large short-circuit currents from flowing. Or the device might not be able to handle the large current that might flow when the output is shorted without being damaged. In those cases:

1. Use a voltmeter to measure the open-circuit at the port of the circuit: $v_{oc} = V_{Th}$.

2. Attach a load resistance, $R_L$ that is small enough so that an appreciable current is flowing. Measure the resulting load voltage, $v_L$.

3. Calculate $R_{Th} = R_L \left( \frac{v_{oc}}{v_L} - 1 \right)$.
Summary

To calculate $V_{Th}$ and $R_{Th}$

1. Using whatever techniques are appropriate, calculate the open-circuit voltage at the port of the circuit: $v_{oc} = V_{Th}$.

2. Connect a short circuit across the output. Using whatever techniques are appropriate, calculate the short-circuit current: $i_{sc} = I_N$.

3. Calculate $R_{Th} = V_{Th} / I_N$.

Alternate method (for circuits that consist only of independent sources and resistors).

1. Using whatever techniques are appropriate, calculate the open-circuit voltage at the port of the circuit: $v_{oc} = V_{Th}$.

2. De-activate all independent sources. Calculate the equivalent resistance as seen from the port. (If dependent sources are present in the circuit, the test generator method can be used to find equivalent resistance. See the equivalent resistance notes to review the test generator technique.)
Example 1

Find the Thevenin and Norton equivalents of the circuit at right, with the port as shown.

Find \( v_{oc} \). Start with a current divider.

\[
\frac{1}{R_2 + R_3} I_S = \frac{1}{6} + \frac{1}{1+3} (24\text{mA}) = 14.4\text{mA}
\]

\[
v_{oc} = i_{R3} R_3 = (14.4\text{mA}) (3\text{k}\Omega) = 43.2\text{V}
\]

Find \( i_{sc} \). Note that \( R_3 \) is shorted out. Use a current divider again.

\[
i_{sc} = \frac{1}{R_2} I_S = \frac{1}{\frac{1}{R_2} + \frac{1}{R_2}} (24\text{mA}) = 20.57\text{mA}
\]

\[
I_N = i_{sc} = 20.57\text{ mA}
\]

\[
V_{Th} = v_{oc} = 43.2\text{ V}
\]
\[ R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{43.2\text{V}}{20.57\text{mA}} = 2.1\text{k}\Omega \]

Alternatively, we could use the short-cut method to find \( R_{Th} \).
De-activating the current source:

\[ R_{eq} = R_3 \parallel (R_1 + R_2) = (3\text{k}\Omega) \parallel (6\text{k}\Omega + 1\text{k}\Omega) = 2.1\text{k}\Omega \]
Example 2

Find the Thevenin and Norton equivalents of the circuit at left, with the port as shown.

Find $v_{oc}$. Use mesh current method.

$$V_S - v_{R1} - v_{R2} - v_{R3} = 0$$
$$v_{R2} - v_{R4} - v_{R5} - v_{R6} = 0$$
$$V_S - R_1i_a - R_2(i_a - i_b) - R_3i_a = 0$$
$$R_2(i_a - i_b) - R_4i_b - R_5i_b - R_6i_b = 0$$

$$i_b = 0.930 \text{ mA}.$$

$$v_{oc} = i_b R_5 = 4.37 \text{ V}.$$
Find $i_{sc}$. Note that $R_5$ is shorted out by the short circuit.

Use equivalent resistance to find $i_s$.

$$R_S = R_1 + R_3 + R_2 \parallel (R_4 + R_6) = 5.75\, \text{k}\Omega$$

$$i_s = \frac{V_S}{R_S} = 3.48\, \text{mA}$$

Use current divider to find $i_{sc}$.

$$i_{sc} = \frac{1}{\frac{1}{R_4+R_6} + \frac{1}{R_2}} \cdot i_s$$

$$i_{sc} = 1.45\, \text{mA}$$

$$R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{4.37\, \text{V}}{1.45\, \text{mA}} = 3.02\, \text{k}\Omega$$
Alternatively, we can use the short-cut method to find $R_{Th}$.

\[ R_{Th} = R_5 \parallel [R_4 + R_6 + R_2 \parallel (R_1 + R_3)] \]

\[ = 3.02 \text{ k}\Omega \]
Example 3

Find the Thevenin and Norton equivalents of the circuit at right, with the port as shown.

Find \( v_{oc} \). Use node voltage.

\[
\begin{align*}
\frac{V_S - v_a}{R_1} &= \frac{v_a}{R_2} + \frac{v_a - v_b}{R_3} \\
\frac{v_a - v_b}{R_3} + \frac{V_S - v_b}{R_5} + I_S &= \frac{v_b}{R_4}
\end{align*}
\]

\[
\begin{align*}
\left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right) v_a - \frac{R_1}{R_3} v_b &= V_S \\
-\frac{R_5}{R_3} v_a + \left(1 + \frac{R_5}{R_3} + \frac{R_5}{R_4}\right) v_b &= V_S + R_5 I_S
\end{align*}
\]

solve: \( v_a = 20 \text{ V}, \ v_b = 40 \text{ V} \).

\( v_{oc} = v_b = 40 \text{ V} \)
Find $i_{sc}$. Use node voltage, again. Note that $R_4$ is shorted out (so ignore it) and node $b$ is shorted to ground, $v_b = 0$.

\[ i_{sc} = i_{R3} + i_{R5} + I_S \]

\[ = \frac{v_a - v_b}{R_3} + \frac{V_S - v_b}{R_5} + I_S \]

\[ = \frac{v_a}{R_3} + \frac{V_S}{R_5} + I_S \]

\[ i_{sc} = \frac{8.57V}{30\Omega} + \frac{30V}{15\Omega} + 4A = 6.28A \]

\[ R_{Th} = \frac{40V}{6.28A} = 6.36\Omega \]

Need $v_a$.

\[ i_{R1} = i_{R2} + i_{R3} \]

\[ \frac{V_S - v_a}{R_1} = \frac{v_a}{R_2} + \frac{v_a}{R_3} \]

\[ v_a = \frac{V_S}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}} = 8.57V \]
Maximum power transfer

Now that we have the ability to model any circuit using a simple Thevenin (or Norton) equivalent, we can answer another important question: How much power can a given circuit supply to an attached load?

Start with the Thevenin equivalent and determine the load resistance that would lead to the maximum amount of power being dissipated in the load.

\[ P_L = \frac{v_{RL}^2}{R_L} = \frac{R_L V_{Th}^2}{(R_L + R_{Th})^2} \]

In the usual way, find the max by setting the derivative to zero and solving.

\[ \frac{dP_L}{dR_L} = 0 \]

\[ \frac{V_{Th}^2}{(R_L + R_{Th})^2} - 2 \frac{R_L V_{Th}^2}{(R_L + R_{Th})^3} = 0 \]

For maximum power to the load

\[ R_L + R_{Th} - 2R_L = 0 \quad \rightarrow \quad R_L = R_{Th} \]

See slide 4 for an example – look at power column in the table.