First-order filters

The general form for the transfer function of a first order filter is:

\[ T(s) = \frac{a_1 s + a_0}{b_1 s + b_0} \]

However, we will typically recast this into a standard form:

\[ T(s) = G_o \cdot \frac{s + Z_o}{s + P_o} \]

There will always be a single pole at \( s = -P_o \). The pole must be real (there is only one, so no complex conjugates are not possible) and it must be negative (for stability). There will always be a zero, which can be at \( s = 0 \), as \( s \to \pm \infty \) (zero at infinity), or somewhere else, \( s = -Z_o \). (Note the zero can have a positive value.) There may be a gain factor, \( G_o \), which might be 1 or smaller (for a passive circuit with a voltage divider) or have a magnitude greater than 1 for an active circuit.

The two most important cases are the zero at infinity, which is a low-pass filter and the zero at zero, which is the high-pass filter.
Low-pass

In the case were \( a_1 = 0 \), we have a low-pass function.

\[
T(s) = \frac{a_o}{b_1s + b_o}
\]

In standard form, we write it as:

\[
T(s) = G_o \cdot \frac{P_o}{s + P_o}
\]

The reason for this form will become clear as we proceed. We will ignore the gain initially and focus on sinusoidal behavior by letting \( s = j\omega \).

\[
\left. \frac{P_o}{s + P_o} \right|_{s = j\omega} = \frac{P_o}{P_o + j\omega}
\]

Re-expressing the complex value in magnitude and phase form:

\[
\frac{P_o}{P_o + j\omega} = \left[ \frac{P_o}{\sqrt{P_o^2 + \omega^2}} \right] \exp (j\theta_{LP}) \quad \theta_{LP} = -\arctan \left( \frac{\omega}{P_o} \right)
\]
By looking at the magnitude expression, we can see the low-pass behavior.

For low frequencies ($\omega \ll P_o$):

$$\frac{P_o}{\sqrt{P_o^2 + \omega^2}} \approx \frac{P_o}{\sqrt{P_o^2}} = 1.$$ 

At high frequencies ($\omega \gg P_o$):

$$\frac{P_o}{\sqrt{P_o^2 + \omega^2}} \approx \frac{P_o}{\sqrt{\omega^2}} = \frac{P_o}{\omega}.$$ 

At low frequencies, the magnitude is 1 (the output is equal to the input) and at high frequencies, the magnitude goes down inversely with frequency, consistent with the notion of a low-pass response.

We can also examine the phase at the extremes.

For low frequencies ($\omega \approx 0$): $\theta_{LP} = - \arctan \left( \frac{\omega}{P_o} \right) \approx 0.$

At high frequencies ($\omega \to +\infty$): $\theta_{LP} = - \arctan \left( \frac{\omega}{P_o} \right) \approx -90^\circ.$
We can use the functions to make the magnitude and phase as frequency response plots.

\[ M = \frac{P_o}{\sqrt{P_o^2 + \omega^2}} \]

\[ \theta_{LP} = -\arctan\left(\frac{\omega}{P_o}\right) \]
Cut-off frequency

Use the standard definition for cut-off frequency, which is the frequency at which the magnitude is down by $\sqrt{2}$ from the value in the pass-band. For our low-pass function, the pass band is at low frequencies, and the magnitude there is 1. (Again, we are “hiding” $G_o$ by assuming that it is unity. If $G_o \neq 1$, then everything is scaled by $G_o$.)

$$M = \frac{1}{\sqrt{2}} = \frac{P_o}{\sqrt{P_o^2 + \omega_c^2}}$$

With a bit of algebra, we find that $\omega_c = P_o$. The cut-off frequency is defined by the pole. Tricky! Thus, in all of our equations, we could substitute $\omega_c$ for $P_o$.

We can also calculate the phase at the cut-off frequency.

$$\theta_{LP} = -\arctan \left( \frac{\omega_c}{P_o} \right) = -45^\circ$$

The cut-off frequency points are indicated in the plots on the previous slide.
To emphasize the importance of the corner frequency in the low-pass function, we can express all the previous results using \( \omega_c \) in place of \( P_o \). On the left are the functions in "standard" form. On the right, the functions are expressed in a slightly different form that is sometimes easier to use.

\[
T_{LP} (s) = G_o \cdot \frac{\omega_c}{s + \omega_c} \quad T_{LP} (s) = \frac{G_o}{1 + \frac{s}{\omega_c}}
\]

\[
T_{LP} (j\omega) = G_o \cdot \frac{\omega_c}{j\omega + \omega_c} \quad T_{LP} (j\omega) = \frac{G_o}{1 + j \left( \frac{\omega}{\omega_c} \right)}
\]

\[
|T_{LP} (j\omega)| = |G_o| \cdot \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}} \quad |T_{LP} (j\omega)| = \frac{|G_o|}{\sqrt{1 + \left( \frac{\omega}{\omega_c} \right)^2}}
\]

\[
\theta_{LP} = - \arctan \left( \frac{\omega}{\omega_c} \right) \quad \theta_{LP} = - \arctan \left( \frac{\omega}{\omega_c} \right)
\]

Again, we could just as easily use real frequency rather than angular frequency. As an exercise: re-express all of the above formulas using \( f \) instead of \( \omega \).
Low-pass filter circuits: simple RC

Resistor and capacitor in series — output taken across the capacitor.

Use a voltage divider to find the transfer function.

\[ V_o(s) = \frac{Z_C}{Z_C + Z_R} V_i(s) \]

\[ T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{\omega_c}{s + \omega_c} \]

Clearly, this is low-pass with \( G_o = 1 \) and \( \omega_c = \frac{1}{RC} \)

The only real design consideration is choosing the \( RC \) product, which then sets the corner frequency.
Low-pass filter circuits: simple RL

Inductor and resistor in series — output taken across the resistor.

Use a voltage divider to find the transfer function.

\[
V_o(s) = \frac{Z_R}{Z_R + Z_L} V_i(s)
\]

\[
T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_R}{Z_R + Z_L} = \frac{R}{R + sL} = \frac{R}{s + \frac{R}{L}} = \frac{\omega_c}{s + \omega_c}
\]

Again, low-pass behavior with \(G_o = 1\), but now with \(\omega_c = \frac{R}{L}\)

As with the previous example choosing the “RL time constant”, we can define the pass-band of this low-pass filter.
Low-pass filter circuits: inverting op amp

\[ Z_2 = R_2 \parallel \frac{1}{sC} = \frac{R_2}{1 + sR_2C} \]

\[ T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{\frac{1}{sR_2C}} = \frac{-R_2}{R_1} \frac{1}{1 + sR_2C} = \left( -\frac{R_2}{R_1} \right) \left( s + \frac{1}{R_2C} \right) \]

Clearly, this is also low-pass with \( G_0 = -\frac{R_2}{R_1} \) and \( \omega_c = \frac{1}{R_2C} \)

Be careful with the extra negative sign in the gain: \(-1 = \exp(j180^\circ)\)

At low frequencies: \(|T| \rightarrow R_2/R_1\), and \(\theta_T \rightarrow 180^\circ (= -180^\circ)\)

At high frequencies: \(|T| \rightarrow (\omega R_1C)^{-1}\), and \(\theta_T \rightarrow +90^\circ (= -270^\circ)\)
Magnitude and phase plots for an active low-pass filter with $R_1 = 1 \, \text{k}\Omega$, $R_2 = 25 \, \text{k}\Omega$, and $C = 6.4 \, \text{nF}$, giving $f_o = 1000 \, \text{Hz}$ and $G_o = -25$ ( $|G_o| = 28 \, \text{dB}$).
Low-pass filter circuits: non-inverting op amp

Note: It might slightly disingenuous to treat this as if it were some new type of filter — we can readily see that it is a simple RC filter cascaded with a simple non-inverting amp. However, it is still a useful circuit.

\[ V_+(s) = \frac{Z_C}{Z_C + Z_R} = \frac{1}{s + \frac{1}{RC}} \quad \text{simple RC} \]

\[ V_o(s) = \left(1 + \frac{R_2}{R_1}\right) V_+(s) \quad \text{non-inverting amp} \]

\[ T(s) = \frac{V_o(s)}{V_i(s)} = \left(1 + \frac{R_2}{R_1}\right) \left(\frac{1}{s + \frac{1}{RC}}\right) \]

Low-pass with \( G_o = \left(1 + \frac{R_2}{R_1}\right) \) and \( \omega_c = \frac{1}{RC} \)
Low-pass filter circuits: another RC

\[ T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_P}{Z_P + Z_{R1}} \]

\[ = \frac{R_2}{1 + sR_2C} \cdot \frac{1}{R_1 + sR_2C + R_1} \]

\[ = \frac{R_2}{R_2 + R_1 + sR_1R_2C} \]

\[ = \left( \frac{R_2}{R_1 + R_2} \right) \frac{1}{R_P} \cdot \frac{1}{s + \frac{1}{R_P}} \]

\[ = G_o \cdot \frac{\omega_c}{s + \omega_c} \]

Low-pass.

\[ G_o = \frac{R_2}{R_1 + R_2} \]

Note the voltage divider. “Gain” < 1.

\[ \omega_c = \frac{1}{R_P C} \]

The corner depends on the parallel combination.

\[ Z_{R1} = R_1 \]

\[ Z_C = \frac{1}{sC} \]

\[ Z_{R2} = R_2 \]

\[ Z_P = Z_{R2} || Z_C \]

\[ R_P = R_1 || R_2 \]

\[ = \frac{R_2 \left( \frac{1}{sC} \right)}{R_2 + \frac{1}{sC}} \]

\[ = \frac{R_2}{1 + sR_2C} \]
High-pass

In the case were \( a_0 = 0 \), we have a high-pass function.

\[
T(s) = \frac{a_1 s}{b_1 s + b_o}
\]

In standard form, we write it as:

\[
T(s) = G_o \cdot \frac{s}{s + P_o}
\]

We will ignore the gain initially (set \( G_o = 1 \)) and focus on sinusoidal behavior by letting \( s = j\omega \).

\[
\left. \frac{s}{s + P_o} \right|_{s=j\omega} = \frac{j\omega}{P_o + j\omega}
\]

Re-expressing the complex value in magnitude and phase form:

\[
\frac{j\omega}{P_o + j\omega} = \left[ \frac{\omega}{\sqrt{P_o^2 + \omega^2}} \right] \exp(j\theta_{HP}) \quad \theta_{HP} = 90^\circ - \arctan\left( \frac{\omega}{P_o} \right)
\]
By looking at the magnitude expression, we can see the high-pass behavior.

For low frequencies ($\omega \ll P_o$):

$$\frac{\omega}{\sqrt{P_o^2 + \omega^2}} \approx \frac{\omega}{\sqrt{P_o^2}} = \frac{\omega}{P_o}.$$ 

At high frequencies ($\omega \gg P_o$):

$$\frac{\omega}{\sqrt{P_o^2 + \omega^2}} \approx \frac{\omega}{\sqrt{\omega^2}} = 1.$$ 

At low frequencies, the magnitude is increasing with frequency, and at high frequencies, the magnitude is 1 (the output is equal to the input). This behavior is consistent with a high-pass response.

We can also examine the phase at the extremes.

For low frequencies ($\omega \ll P_o$): $\theta_{HP} = 90^\circ - \arctan\left(\frac{\omega}{P_o}\right) \approx 90^\circ.$

At high frequencies ($\omega \gg P_o$): $\theta_{HP} = 90^\circ - \arctan\left(\frac{\omega}{P_o}\right) \approx 0^\circ.$
We can use the functions to make the magnitude and phase as frequency response plots.

\[ M = \frac{\omega}{\sqrt{P_o^2 + \omega^2}} \]

\[ \theta_{HP} = 90^\circ - \arctan \left( \frac{\omega}{P_o} \right) \]
**Cut-off frequency**

Use the standard definition for cut-off frequency, which is the frequency at which the magnitude is down by $\sqrt{2}$ from the value in the pass-band. For our high-pass function, the pass band is at high frequencies, and the magnitude there is 1. (Again, we are “hiding” $G_o$ by assuming that it is unity. If $G_o \neq 1$, then everything is scaled by $G_o$.)

$$M = \frac{1}{\sqrt{2}} = \frac{\omega}{\sqrt{P_o^2 + \omega_c^2}}$$

With a bit of algebra, we find that $\omega_c = P_o$. The same result as for low-pass response, except that pass-band is above the cut-off frequency in this case. Once again, we see the importance of the poles in determining the behavior of the transfer functions.

We can calculate the phase at the cut-off frequency.

$$\theta_{HP} = 90^\circ - \arctan \left( \frac{\omega_c}{P_o} \right) = 45^\circ$$

The cut-off frequency points are indicated in the plots on the previous slide.
To emphasize the importance of the corner frequency in the high-pass function, we can express all the previous results using $\omega_c$ in place of $P_o$.

\[
\frac{G_o}{\sqrt{2}} = \frac{G_o \cdot P_0}{\sqrt{\omega^2 + P_0^2}} \quad \rightarrow \quad P_0 = \omega_c
\]

The corner frequency is the value of the pole.

\[
T_{HP} (s) = G_o \cdot \frac{s}{s + \omega_c}
\]

\[
T_{HP} (j\omega) = G_o \cdot \frac{j\omega}{j\omega + \omega_c}
\]

\[
|T_{LP} (j\omega)| = G_o \cdot \frac{\omega}{\sqrt{\omega^2 + \omega_c^2}}
\]

\[
\theta_{HP} = \arctan \left( \frac{\omega}{0} \right) - \arctan \left( \frac{\omega}{\omega_c} \right)
= 90^\circ - \arctan \left( \frac{\omega}{\omega_c} \right)
\]

Exercise: Re-express all of the above formulas using $f$ instead of $\omega$. 
High-pass filter circuits: simple RC

Capacitor and resistor in series — output taken across the resistor.

Use a voltage divider to find the transfer function.

\[ V_o(s) = \frac{Z_R}{Z_C + Z_R} V_i(s) \]

\[ T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{sC}} = \frac{s}{s + \frac{1}{RC}} = \frac{s}{s + \omega_c} \]

Clearly, this is high-pass with \( G_o = 1 \) and \( \omega_c = \frac{1}{RC} \)
High-pass filter circuits: simple RL

\[ Z_R = R \]

\[ V_i(s) \quad + \quad Z_L \quad + \quad V_o(s) \quad - \]

\[ Z_L = sL \]

\[ V_o(s) = \frac{Z_L}{Z_L + Z_R} V_i(s) \]

\[ T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_L}{Z_L + Z_R} = \frac{sL}{sL + R} = \frac{s}{s + \frac{R}{L}} = \frac{s}{s + \omega_c} \]

Again, high-pass with \( G_o = 1 \) but with \( \omega_c = \frac{R}{L} \)
High-pass filter circuits: inverting op amp

\[ Z_1 = R_1 + \frac{1}{sC} \]

\[ T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{R_1 + \frac{1}{sC}} = \left(-\frac{R_2}{R_1}\right) \frac{s}{s + \frac{1}{R_1C}} \]

We see that this is also high-pass with \( G_o = -\frac{R_2}{R_1} \) and \( \omega_c = \frac{1}{R_1C} \)

The same comments about the phase apply here: the –1 in the gain factor introduces an extra 180° (or –180°) of phase.
High-pass filter circuits: non-inverting op amp

Again, this is simple RC high-pass cascaded with a non-inverting amp.

\[
V_+ (s) = \frac{Z_R}{Z_C + Z_R} = \frac{s}{s + \frac{1}{RC}} \quad \text{simple RC high pass}
\]

\[
V_o (s) = \left(1 + \frac{R_2}{R_1}\right) V_+ (s) \quad \text{non-inverting amp}
\]

\[
T(s) = \frac{V_o(s)}{V_i(s)} = \left(1 + \frac{R_2}{R_1}\right) \left(\frac{s}{s + \frac{1}{RC}}\right)
\]

High-pass with \( G_0 = \left(1 + \frac{R_2}{R_1}\right) \) and \( \omega_c = \frac{1}{RC} \)
High-pass filter circuits: another RC

\[ T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_{R2}}{Z_{R2} + Z_{R1} + Z_C} \]

\[ = \frac{R_2}{R_2 + R_1 + \frac{1}{sC}} \]

\[ = \left( \frac{R_2}{R_1 + R_2} \right) s + \frac{s}{\frac{1}{(R_1+R_2)C}} \]

\[ = G_o \cdot \frac{s}{s + \omega_c} \]  \[ \text{High-pass.} \]

\[ G_o = \frac{R_2}{R_1 + R_2} \]  \[ \text{Note the voltage divider. “Gain” < 1.} \]

\[ \omega_c = \frac{1}{(R_1 + R_2)C} \]  \[ \text{The corner depends on the series combination.} \]
Case study: inductors can be trouble

Cheap inductors can have a relatively large series resistance. This *parasitic* resistance can cause trouble in certain circumstances.

\[ T(s) = \frac{s}{s + \frac{R}{L}} = \frac{s}{s + \omega_0} \]

As usual, zero at \( s = 0 \), pole at \( s = -\omega_c \).

\[ T(s) = \frac{sL + R_s}{sL + R_s + R_1} = \frac{s + \frac{R_s}{L}}{s + \frac{R_s + R_1}{L}} = \frac{s + \omega_Z}{s + \omega_p} \]

Pole and zero are both shifted!
\[ T(j\omega) = \frac{\omega_Z + j\omega}{\omega_P + j\omega} \]

\[ |T| = \frac{\sqrt{\omega_Z^2 + \omega^2}}{\sqrt{\omega_P^2 + \omega^2}} \]

\[ \theta_T = \arctan \left( \frac{\omega}{\omega_Z} \right) - \arctan \left( \frac{\omega}{\omega_P} \right) \]

Effect of inductor parasitic resistance on high-pass filter:
\[ L = 0.027 \text{ H}, \quad R_1 = 1 \text{ k}\Omega, \quad \text{and} \quad R_s = 60 \text{ \Omega}. \]

The frequency responses for both magnitude and phase are quite different.