Solving circuits directly with Laplace

The Laplace method seems to be useful for solving the differential equations that arise with circuits that have capacitors and inductors and sources that vary with time (steps and sinusoids.) The approach has been:

1. Analyze the circuit in the time domain using familiar circuit analysis techniques to arrive at a differential equation for the time-domain quantity of interest (voltage or current).

2. Perform a Laplace transform on differential equation to arrive a frequency-domain form of the quantity of interest.

3. Solve the frequency-domain algebra expression.

4. Transform back to the time-domain.

Might it possible to change the order of the steps? Could we transform the circuit into the frequency domain and then use circuit techniques to find the desired voltage or current? Might this is approach be easier than solving differential equations?

Not surprisingly, the answer to all three questions is “Yes!”
Frequency domain impedances

In order to transform a circuit, we need frequency-domain descriptions of all of the components in the circuit. We already know how to transform the commonly used step and sinusoidal sources. We need to consider resistors, inductors, and capacitors to see the form of the current-voltage relationships in the frequency domain. Apply the Laplace transform to the $i$-$v$ equations directly.

\[ i_R(t) \quad \rightarrow \quad + v_R(t) - \]
\[ v_R(t) = R \cdot i_R(t) \]
\[ V_R(s) = R \cdot I_R(s) \]
\[ \frac{V_R}{I_R} = R \]
\[ I_R(s) \quad \rightarrow \quad + V_R(s) - \]

\[ i_C(t) \quad \rightarrow \quad + v_C(t) - \]
\[ i_C(t) = C \frac{dv_C(t)}{dt} \]
\[ I_C(s) = C \cdot s \cdot V_C(s) \]
\[ \frac{V_C}{I_C} = \frac{1}{sC} \]
\[ V_C(s) \quad \rightarrow \quad + V_C(t) - \]

\[ i_L(t) \quad \rightarrow \quad + v_L(t) - \]
\[ v_L(t) = L \frac{di_L(t)}{dt} \]
\[ V_L(s) = L \cdot s \cdot I_L(s) \]
\[ \frac{V_L}{I_L} = sL \]
\[ I_L(s) \quad \rightarrow \quad + V_L(s) - \]
For the resistor, the frequency domain relationship is exactly the same as the time domain. (Ohm’s Law is not time-dependent, so this is not a surprise.) For the inductor and capacitor, the frequency domain relation is actually simpler. All three components can be treated with a simple “Ohm’s-Law-like” $i-v$ equation:

$$V(s) = Z \cdot I(s)$$

where $Z$ is known as the “impedance”, with units of ohms ($\Omega$).

$$Z_R = R$$

$$Z_C = \frac{1}{sC}$$

$$Z_L = sL$$

$Z_C$ and $Z_L$ depend on frequency, but for a given frequency, they are constants. They are complex constants (since $s$ is complex), but the frequency domain relationships are exactly like those of the resistor: voltage is equal to a constant multiplied by the current. This means that the circuit in the frequency domain can be solved using all of the methods that we learned for circuits with sources and resistors at the very beginning of EE 201.
Note that this very similar to the complex impedance defined when doing AC analysis in EE 201. (In fact, they are identical, if we restrict the complex frequency to just imaginary values, $s = j\omega$. This takes us right back to AC analysis.)

All of the familiar techniques learned in 201 apply in the frequency domain, as well:

- equivalent resistances (now equivalent impedances)
- voltage / current dividers *
- source transformations
- node voltages *
- mesh currents
- superposition

* indicates techniques that are applicable in both the time and frequency domains.
Now, with the approach of transforming the circuit into the frequency domain using impedances, the Laplace procedure becomes:

1. Transform the circuit. Use the Laplace transform version of the sources and the other components become impedances.

2. Solve the circuit using any (or all) of the standard circuit analysis techniques to arrive at the desired voltage or current, expressed in terms of the frequency-domain sources and impedances.

3. Transform back to the time-domain. (If needed.)

The following examples illustrate the method.
Example

Find $v_c(t)$ for the circuit below. The input is a step function, $v_i = V_f \cdot u(t)$

Use a voltage divider:

$$V_C(s) = \frac{Z_C}{Z_C + Z_R} V_i(s)$$

$$= \left( \frac{\frac{1}{sC}}{\frac{1}{sC} + R} \right) \frac{V_f}{s}$$

$$= \left( \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) \frac{V_f}{s}$$

That’s it. Transform back to the time domain, if desired.
Example

Find $i_L(t)$ for the circuit below. The input is a step function, $i_i(t) = I_f \cdot u(t)$

Use a current divider:

$$I_L(s) = \frac{1}{Z_L} \frac{1}{Z_L + \frac{1}{Z_R}} I_{in}(s)$$

$$= \left( \frac{1}{sL} \right) \frac{1}{sL + \frac{1}{R}} I_f$$

$$= \left( \frac{R}{sL + \frac{R}{L}} \right) \frac{I_f}{s}$$

Transform back, if needed.
Example

Find $v_c(t)$ for the *RLC* circuit below.
The input is a step function, $v_i(t) = V_f \cdot u(t)$.

Once again, use a divider:

$$V_C(s) = \frac{Z_C}{Z_C + Z_R + Z_L} V_i(s)$$

$$= \left( \frac{\frac{1}{sC}}{\frac{1}{sC} + R + sL} \right) \frac{V_f}{s}$$

$$= \left( \frac{\frac{1}{LC}}{s^2 + \frac{R}{L} s + \frac{1}{LC}} \right) \frac{V_f}{s}$$

So easy.
Example

Find $v_o(t)$ for the op amp circuit below. The input is a step function, $v_i(t) = V_f \cdot u(t)$. Using equivalent resistances was an important “short-cut” tool in 201. We can extend to the idea of equivalent impedances. Often, we can make the analysis for a circuit quite easy with the right impedance combinations. Also, op amp rules carry over directly to the frequency domain.

\[
V_i(s) = \frac{V_f}{s}
\]

\[
V_o(s) = \left( -\frac{Z_2}{Z_1} \right) V_i(s)
\]

\[
V_o(s) = -\left( \frac{R_2}{R_1 + \frac{1}{sC}} \right) \frac{V_f}{s} = \left( \frac{-\frac{R_2}{R_1}}{s + \frac{1}{R_1C}} \right) \frac{V_f}{s}
\]

In the frequency domain, we recognize the amp in the inverting configuration.

$Z_1$ is the series combination of the impedances of $R_1$ and $C$.

\[
Z_1 = R_1 + \frac{1}{sC}
\]

$Z_2 = R_2$
Example

Find $i_1(t)$ for the circuit below. The input is a step function, $v_i(t) = V_f \cdot u(t)$.

\[ I_1(s) = \frac{V_i(s)}{Z_1 + Z_2} \]

\[ = \left( \frac{\frac{V_f}{s}}{R_1 + sL + \frac{R_2}{1 + sR_2C}} \right) \]

\[ = \left( \frac{1 + sR_2C}{s^2R_2LC + sR_1R_2C + sL + R_1} \right) \frac{V_f}{s} \]

Exercise: Confirm that the quantity in parentheses has units of $\Omega^{-1}$. 