Impedance and AC circuit analysis

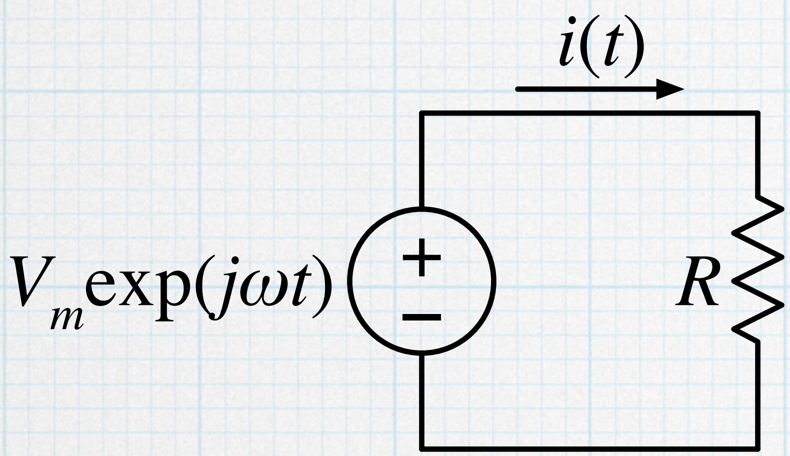
So far, we have seen that

1. We are willing to ignore the transient portion in the analysis of AC circuits, eliminating more than half of the mathematical drudgery inherit in solving differential equations from scratch. (Sinusoidal steady-state analysis.)

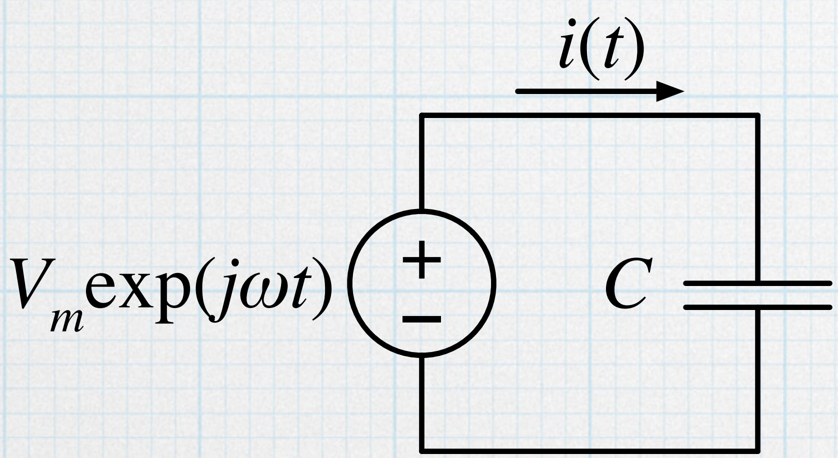
2. Using complex exponentials to describe the sinusoidal source is much more effective than slashing around with sines and cosines. The effect of using complex sources is to cause all of the voltages and currents in the circuit to be complex. These complex descriptions contain the magnitudes and phase angles that are the the most interesting aspects of AC circuits. By expressing the complex numbers in magnitude/phase form, the key parts of the sinusoidal voltages and currents pop right out.

Now we are ready to invoke the last simplification that we will use in AC circuits — the notion of impedances.

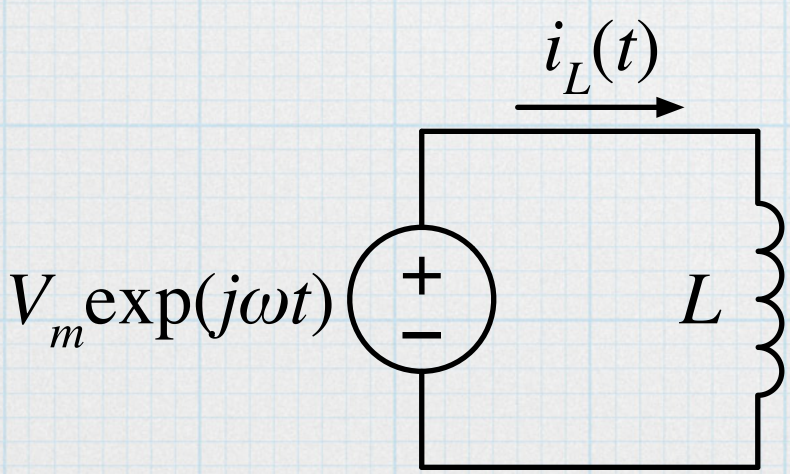
Apply a complex voltages sources to individual resistors, capacitors, and inductors.



$$i(t) = \frac{V_m e^{j\omega t}}{R} = \frac{v_R(t)}{R} \qquad \frac{v_R(t)}{i_R(t)} = R$$



$$i_C(t) = C\frac{d\left(V_m e^{j\omega t}\right)}{dt} = j\omega C\left[v_C(t)\right] \qquad \frac{v_C(t)}{i_C(t)} = \frac{1}{j\omega C}$$

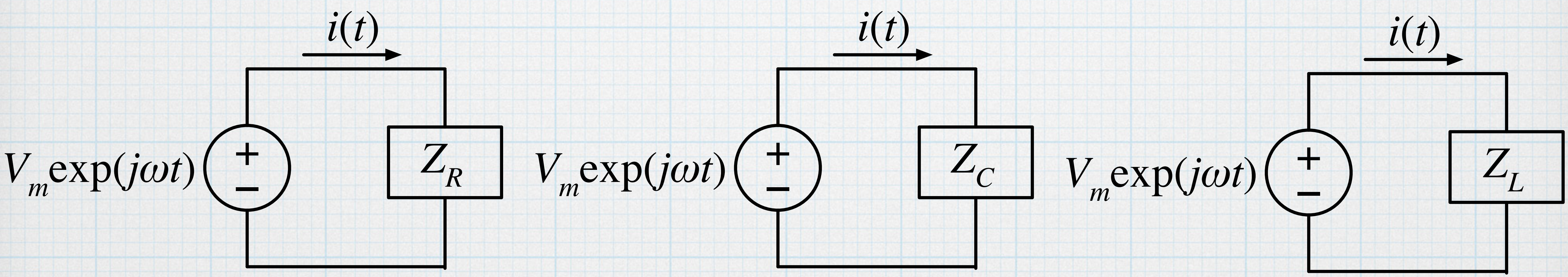


$$i_L(t) = \frac{1}{L}\int V_m e^{j\omega t} dt = \frac{v_L(t)}{j\omega L} \qquad \frac{v_L(t)}{i_L(t)} = j\omega L$$

In each case, when a sinusoidal voltage is applied a sinusoidal current results. And in each case, the ratio of the of sinusoidal voltage to the sinusoidal current is a constant. For the resistor, the ratio is just the value of the resistance. (No big surprise here, Ohm's law still applies.)

For the capacitor and inductor, the ratio is a purely imaginary number that depends of the value of the component and the frequency.

For the three cases, we can generalize and call all of the ratios *impedances*. The usual symbol for impedance is *Z*.



$$\frac{v(t)}{i(t)} = Z_R \qquad \frac{v(t)}{i(t)} = Z_C \qquad \frac{v(t)}{i(t)} = Z_L$$

$$Z_R = R \qquad Z_C = \frac{1}{j\omega C} \qquad Z_L = j\omega L$$

This observation, that for each of the principle components (*R*, *L*, *C*) the ratio of sinusoidal voltage to sinusoidal current is a constant, has a profound implication. It means that we don't have to come up with differential equations and solve them. We can apply the techniques that we learned using just resistors to sinusoidal circuits.

For example, recall the node-voltage technique. We used Kirchoff's current laws to relate the current flowing into and out of node. Then we used Ohm's Law to related the resistor currents to the node voltages on either side of the resistors. The result was a set of equations that could be solved to find the node voltages.

In the sinusoidal case, we use Kirchoff's current law to relate the sinusoidal currents at a node. (KCL is fundamental — it applies to sinusoids, too.) Then we can use the impedance relations to relate the sinusoidal currents at each node to the sinusoidal node voltages. The impedances work just like resistors in our earlier efforts, except that they are complex. The derivative dependencies of the capacitor and inductor are built into the impedances.

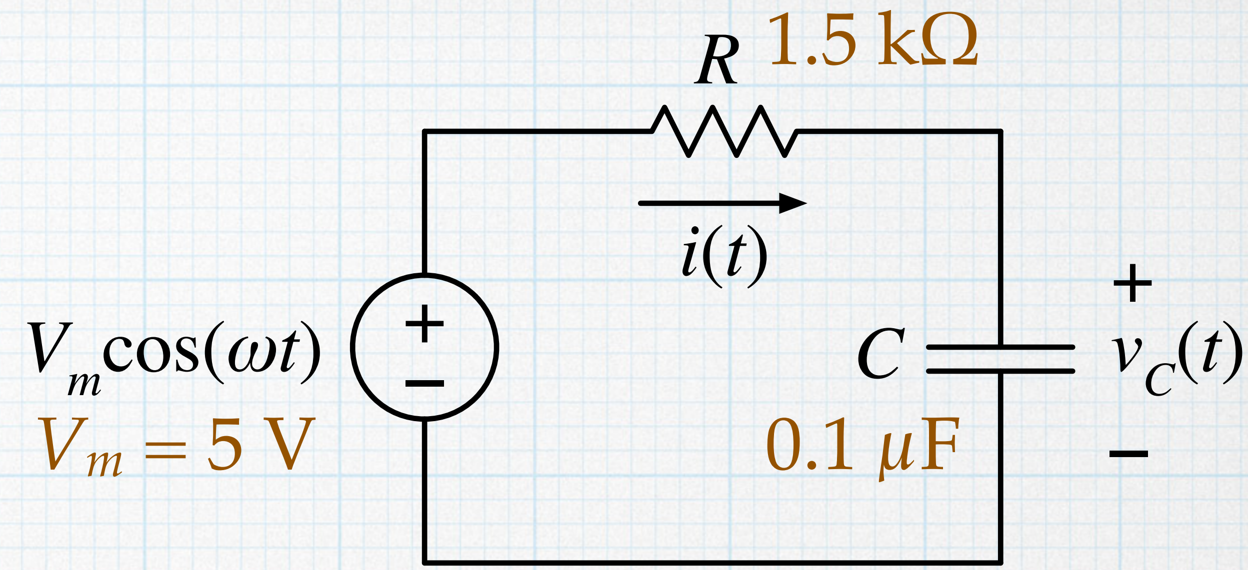
So we can use the impedances with Kirchoff's Laws to come up with the circuit equations that are not differential equations. (Yay!!) However, the relating circuit equations will involve complex currents, voltages, and impedances. (Boo!!)

The procedure for solving AC circuits using impedances is:

1. Convert the sources to exponential sinusoids.
2. Convert the components to impedances.
3. Using the usual collection of tools (equivalent *impedances*, dividers, source transformations, node voltage, mesh current, superposition) find the required *complex* voltages and currents in the current.
4. Grind through the complex algebra to find the required voltages and currents.
5. When finished, express the complex voltages and currents in magnitude and phase form. The magnitude/phase form tells us exactly what we will see if the we built the circuit in the lab and used an oscilloscope to view the sinusoidal wave forms.

Example 1: RC circuit

To test this newest approach, we will rework the simple *RC* circuit from the previous two set of notes to find v_C. First convert the circuit over to it's complex form, a complex sinusoid for the source and impedances for the resistor and capacitor.



$\omega = 6660$ rad/s

($f = 1060$ Hz. $T = 0.943$ ms)

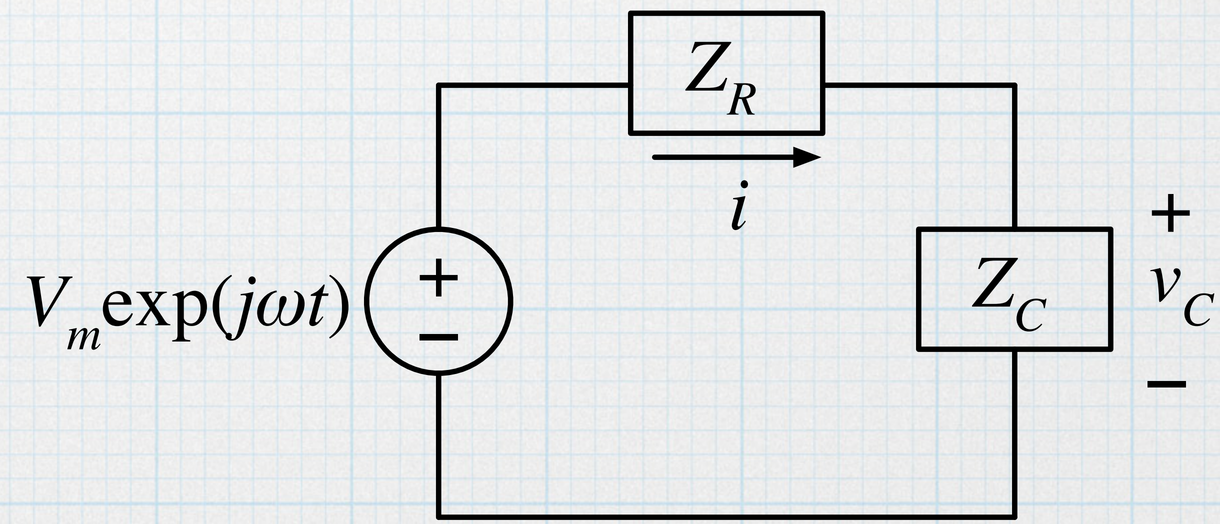
Using impedances, we can treat this a simple voltage divider.

$$v_C = \frac{Z_C}{Z_C + Z_R} V_S$$

Insert the forms of impedances

$$v_C = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} V_S$$

$$= \frac{V_S}{1 + j\omega RC}$$



$$v_C = \frac{V_m e^{j\omega t}}{1 + j\omega RC}$$

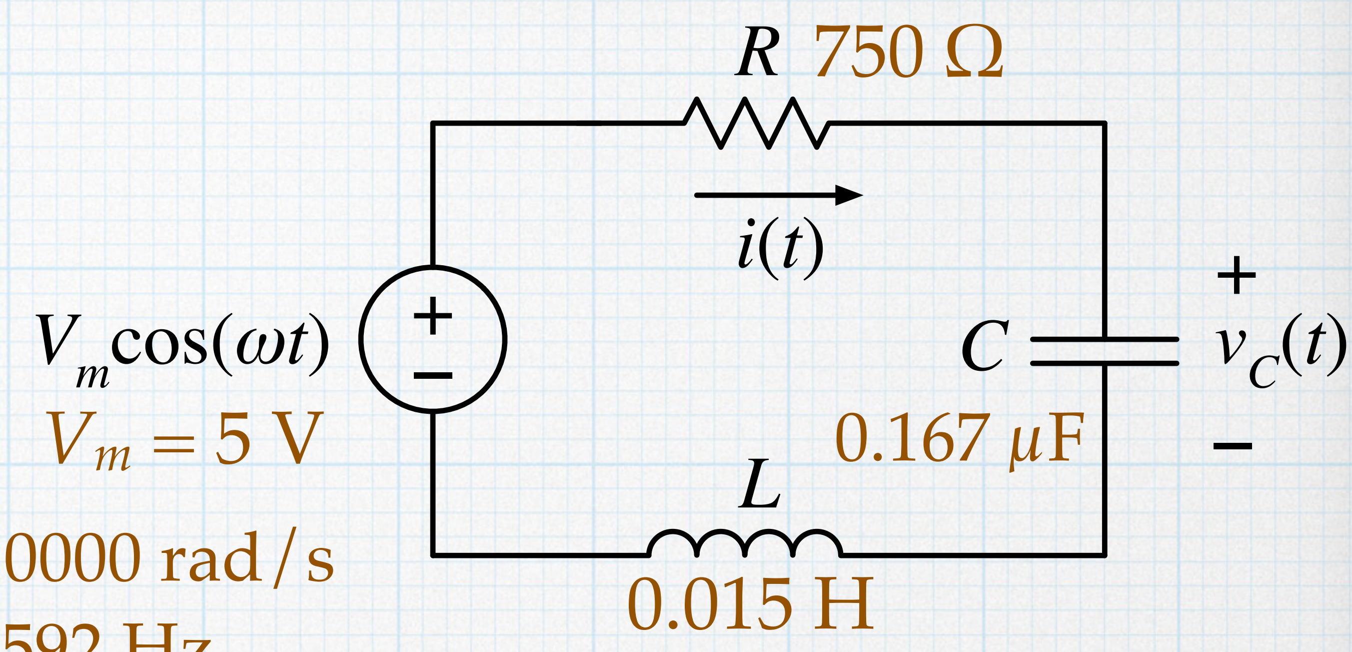
$$v_C = M e^{j(\omega t + \theta)} \qquad M = \frac{V_m}{\sqrt{1 + (\omega RC)^2}} \qquad \theta = -\arctan(\omega RC)$$

$$= 3.54\ \mathrm{V} \qquad = -45°$$

Identical to the previous results. The circuit analysis is straight-forward and no differential equations. The only thing is that is tricky is the complex analysis.

A second example

Let's apply the complex approach to the *RLC* example done earlier.

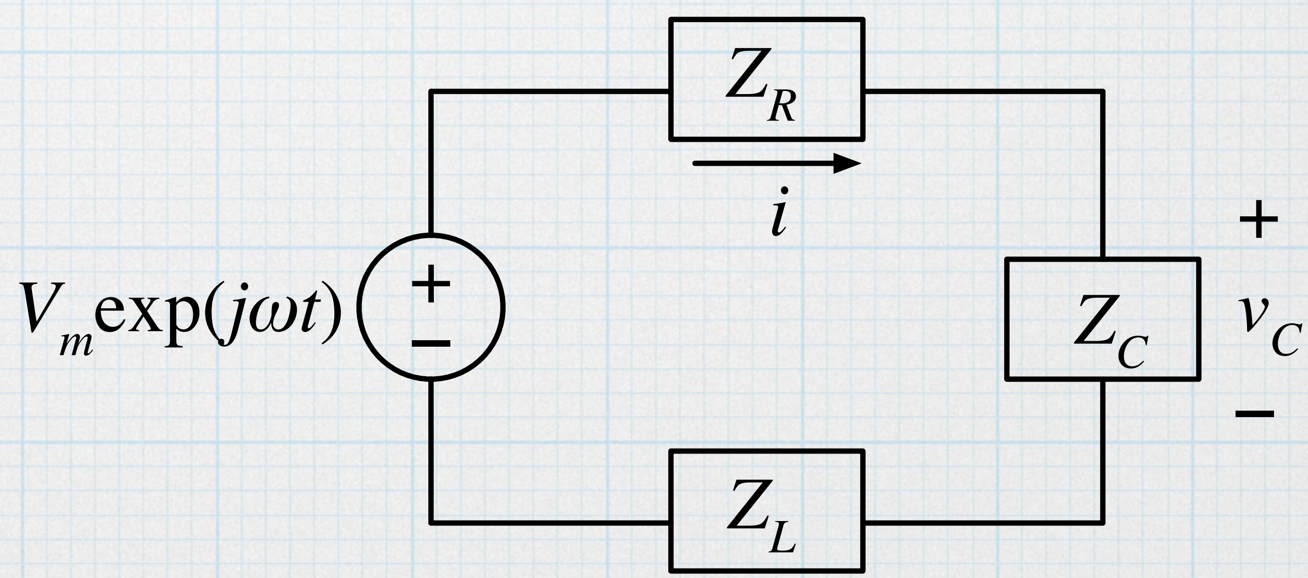


$\omega = 10000$ rad/s
(f = 1592 Hz
T = 0.628 ms)

Once again, the impedances form a voltage divider.

$$v_C = \frac{Z_C}{Z_C + Z_L + Z_R} V_S$$

Insert the impedances



$$v_C = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L + R} V_S$$

$$= \frac{V_S}{1 - \omega^2 LC + j\omega RC}$$

$$v_C = \frac{V_m e^{j\omega t}}{1 - \omega^2 LC + j\omega RC}$$

$$v_C = M e^{j(\omega t + \theta)}$$

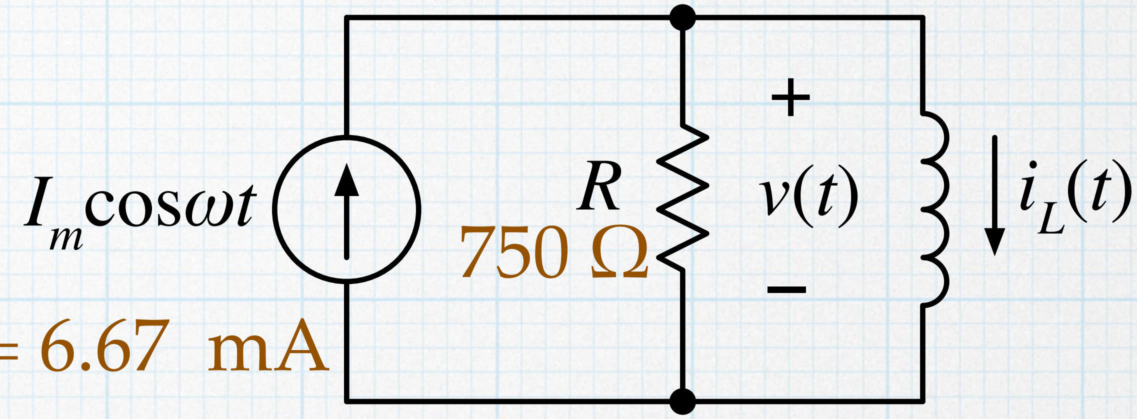
$$M = \frac{V_m}{\sqrt{\left[1 - (\omega LC)^2\right]^2 + (\omega RC)^2}} \quad = 3.43 \text{ V}$$

$$\theta = -\arctan\left(\frac{\omega RC}{1 - \omega^2 LC}\right)$$

$$= -59°$$

Example 3: RL circuit

To finish off this section, we will use the impedance approach with the *RL* circuit.



$I_m = 6.67$ mA

$\omega = 50000$ rad/s

$L = 15$ mH

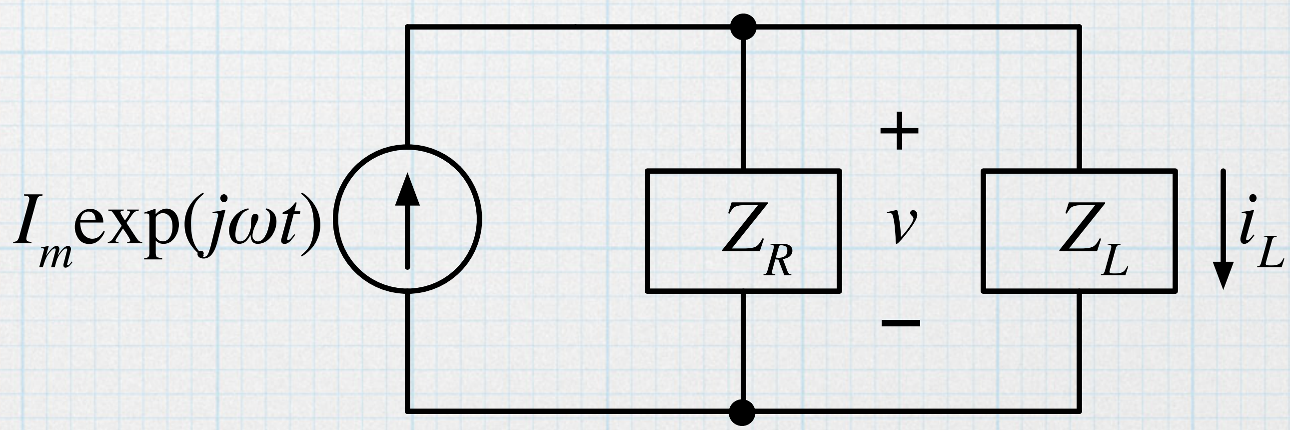
($f = 7958$ Hz. $T = 0.126$ ms)

Now we can use a current divider

$$i_L = \frac{\frac{1}{Z_L}}{\frac{1}{Z_L} + \frac{1}{Z_R}} I_S$$

$$= \frac{\frac{1}{j\omega L}}{\frac{1}{j\omega L} + \frac{1}{R}} I_S$$

$$= \frac{1}{1 + j\frac{\omega L}{R}} I_S$$



Once again, jump straight to the steady-state equation,

$$i_L = \frac{I_m e^{j\omega t}}{1 + j\frac{\omega L}{R}}$$

$$i_L = Me^{j(\omega t + \theta)}$$

$$M = \frac{I_m}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \quad = -4.71\,\text{mA}$$

$$\theta = -\arctan\left(\frac{\omega L}{R}\right) \quad = -45°$$