The node voltage method

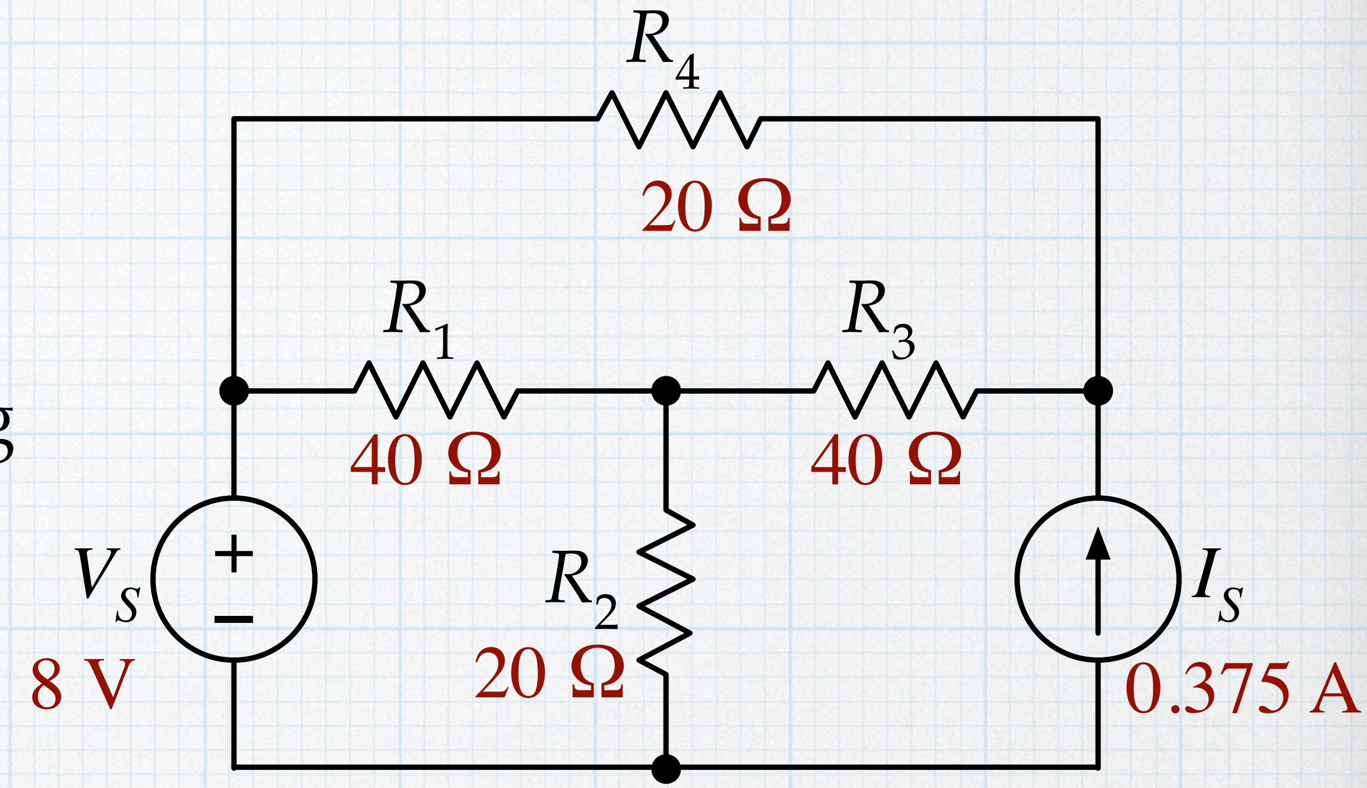
- Equivalent resistance
- Voltage / current dividers
- Source transformations
- Node voltages
- Mesh currents
- Superposition

Not every circuit lends itself to “short-cut” methods. Sometimes we need a formal approach that does not rely on using a trick. The node-voltage is the first (and maybe most used) of our three formal methods.

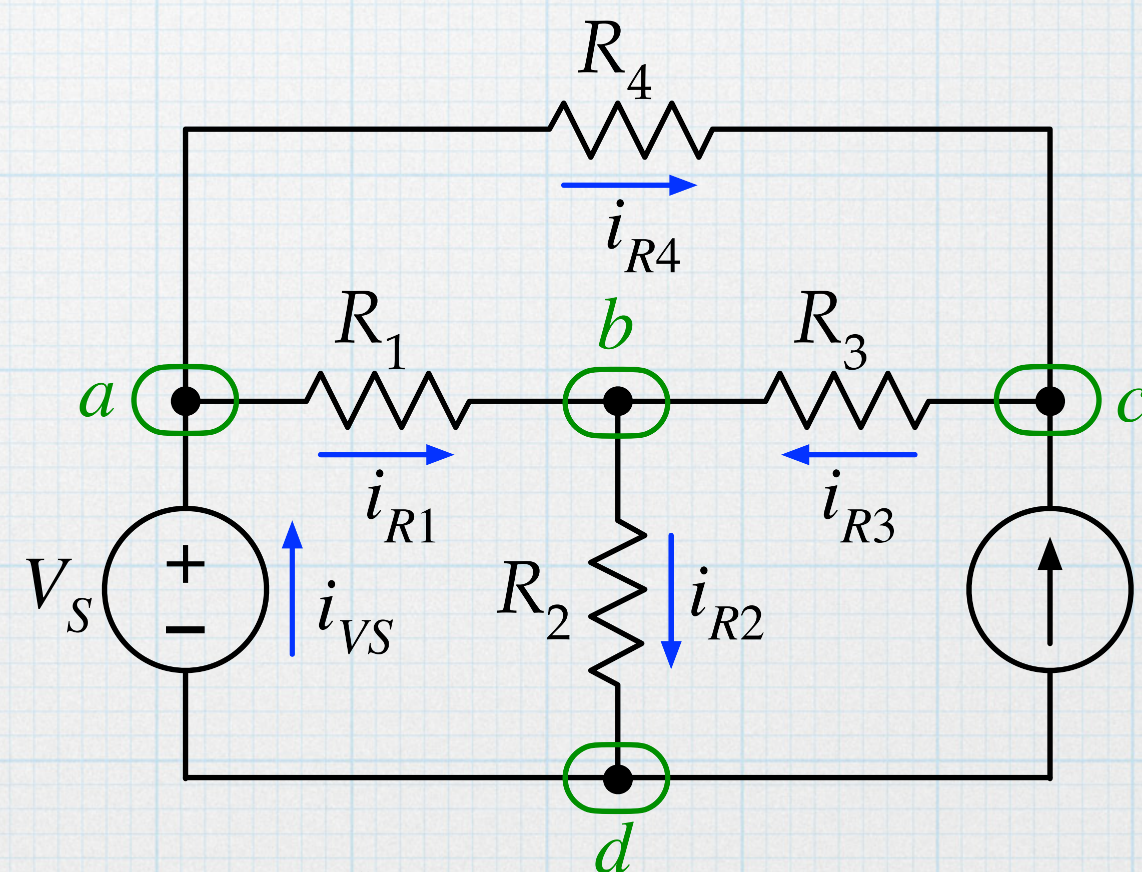
The node-voltage method is a systematic approach for deriving a set of simultaneous equations that can be solved to find the voltage at each node of the circuit. Once the node voltages are known, all currents and powers in the circuit follow easily. The method is identical for any size circuit, although the math will be messier for bigger circuits since the number of simultaneous equation scales with the number of nodes.

An example

Consider the circuit at right. It looks easy enough, but we are quickly disappointed when we find that no short-cut methods will help in trying to solve it. We must go back to Kirchoff's Laws.



Let's poke at this a bit using KCL. First identify all of the nodes — there are four in this case. Label the currents in each branch — the directions can be chosen arbitrarily. Then write KCL equations at the four nodes.



a. $i_{VS} = i_{R1} + I_{R4}$

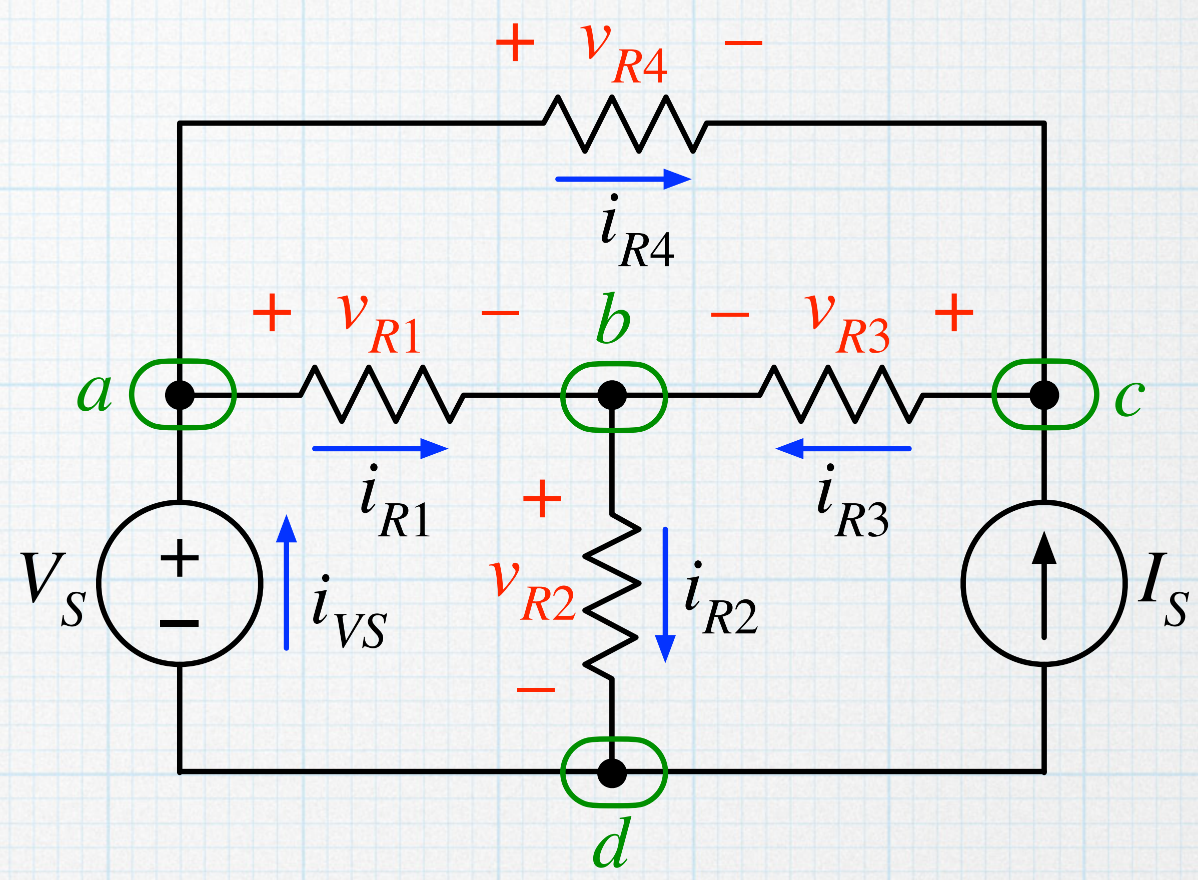
b. $i_{R1} + i_{R3} = i_{R2}$

c. $i_{R4} + I_S = i_{R3}$

d. $i_{R2} = I_S + i_{VS}$

There are four equations relating five unknown currents — not good.

We can use Ohm's law to write resistor currents in terms of the resistor voltages. (Pay attention to polarities. For the resistors, the voltage polarities must match the chosen current directions.)



a. $i_{VS} = \dfrac{v_{R1}}{R_1} + \dfrac{v_{R4}}{R_4}$

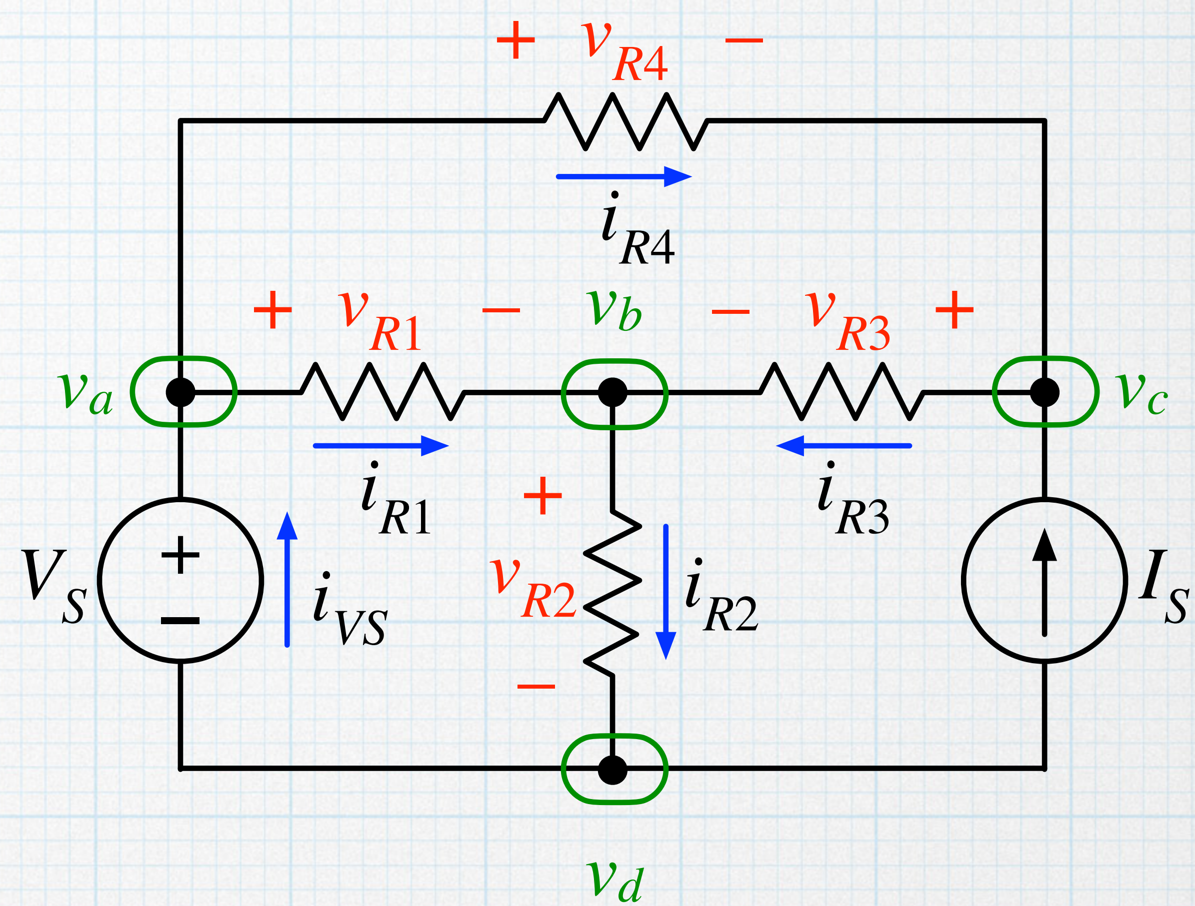
b. $\dfrac{v_{R1}}{R_1} + \dfrac{v_{R3}}{R_3} = \dfrac{v_{R2}}{R_2}$

c. $\dfrac{v_{R4}}{R_4} + I_S = \dfrac{v_{R3}}{R_3}$

d. $\dfrac{v_{R2}}{R_2} = I_S + i_{VS}$

This doesn't improve the situation — there are still four equations relating five unknowns — four resistor voltages and the current flowing through the voltage source.

We can take another step and assign a voltage to each node. (Recall that our definition of a node is a point of connection between components, and the node has a single voltage.) We can then write the resistor voltages as differences between the node voltages.



a. $i_{VS} = \dfrac{v_a - v_b}{R_1} + \dfrac{v_a - v_c}{R_4}$

b. $\dfrac{v_a - v_b}{R_1} + \dfrac{v_c - v_b}{R_3} = \dfrac{v_b - v_d}{R_2}$

c. $\dfrac{v_a - v_c}{R_4} + I_S = \dfrac{v_c - v_b}{R_3}$

d. $\dfrac{v_b - v_d}{R_2} = I_S + i_{VS}$

$v_{R1} = v_a - v_b$

$v_{R2} = v_b - v_d$

$v_{R3} = v_c - v_b$

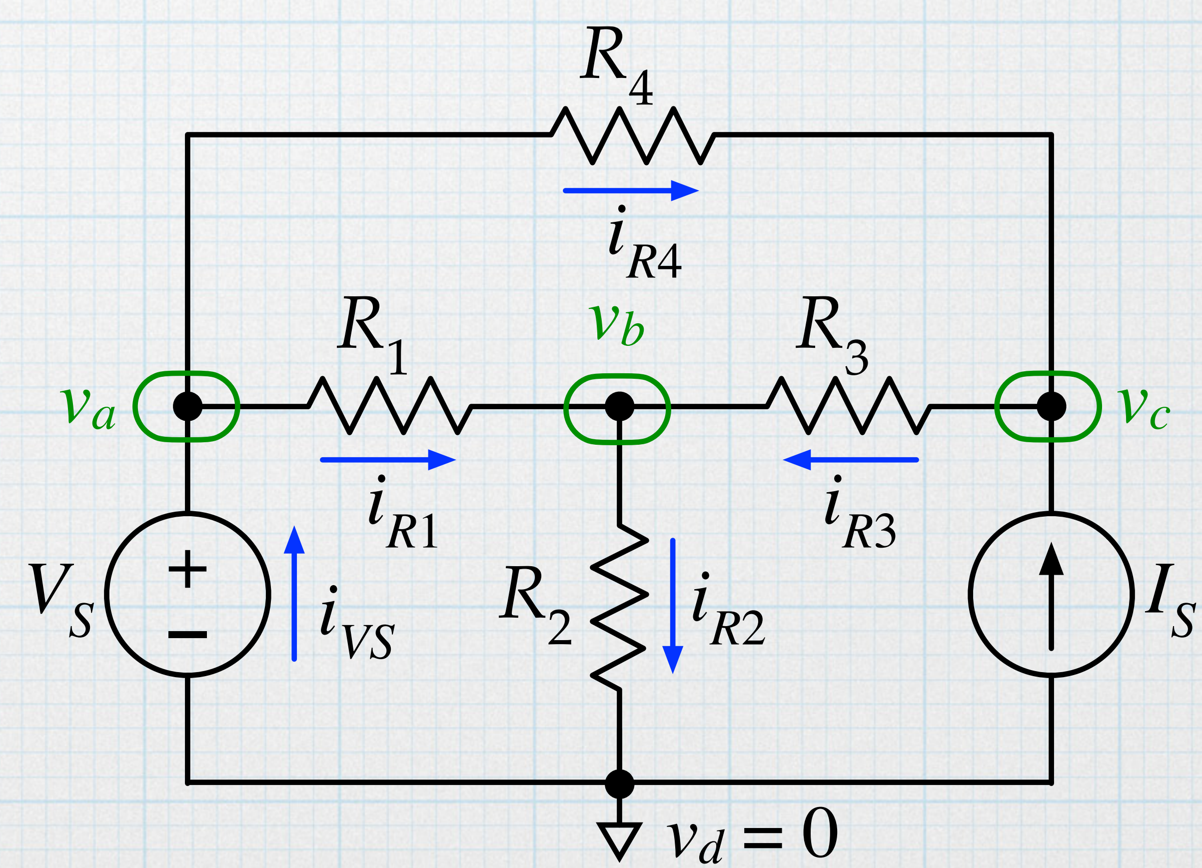
$v_{R4} = v_a - v_c$

However, it seems that we are going in circles, since there are still 5 unknowns — v_a, v_b, v_c, v_d, and i_{VS}. Basically, we have just been changing names. But we are ready for a crucial step.

Voltage, like energy, is a relative quantity — only *differences* are important. The absolute values of v_a, v_b, v_c, and v_d do not matter — only the differences, $v_a - v_b$, $v_a - v_c$, etc. are important, as we saw in the previous set of equations. This means that we can *assign* a voltage value to one node, and then all other node voltages can be defined *with respect to* that chosen node voltage. We could assign any voltage that we want, but an obvious value would be 0 V. When a particular node is chosen to have "0 volts", we call it the *ground* node.

We are free to choose any of the nodes in the circuit to be the ground. We will see in the examples to follow that some choices are better than others, but, at least initially, each node is an equally viable ground.

In this example, we will choose node d to be ground, and so, by our definition $v_d = 0$. Once we have chosen a node to serve as ground, we denote that in the circuit with the ground symbol, as shown at right.



Choosing a ground node to serve as a voltage reference has two significant effects on the set of equations that describe the circuit. The first is that, since the voltage at d was assigned to be zero, it is no longer "unknown" and our math problem reduces to a set of four equations with four unknowns — v_a, v_b, v_c, and i_{VS}. The set can be solved!

a. $i_{VS} = \dfrac{v_a - v_b}{R_1} + \dfrac{v_a - v_c}{R_4}$

b. $\dfrac{v_a - v_b}{R_1} + \dfrac{v_c - v_b}{R_3} = \dfrac{v_b}{R_2}$

c. $\dfrac{v_a - v_c}{R_4} + I_S = \dfrac{v_c - v_b}{R_3}$

d. $\dfrac{v_b}{R_2} = I_S + i_{VS}$

We could just get to work and solve these equations, but we can do more to make the job easier.

The second effect is that, by defining node d as ground, we also immediately know the value of v_a. By the definition of a voltage source, $v_a = v_d + V_S = 0 + V_S = V_S$. So we also know v_a. The four equations now have only 3 unknowns — we have the luxury of choosing which equations to solve to find the remaining unknown quantities.

a. $$i_{VS} = \frac{V_S - v_b}{R_1} + \frac{V_S - v_c}{R_4}$$

b. $$\frac{V_S - v_b}{R_1} + \frac{v_c - v_b}{R_3} = \frac{v_b}{R_2}$$

c. $$\frac{V_S - v_c}{R_4} + I_S = \frac{v_c - v_b}{R_3}$$

d. $$\frac{v_b}{R_2} = I_S + i_{VS}$$

In examining the set of equations, we see that the middle two equations depend only on the node voltages, v_b and v_c. So we could solve just those two equations to find v_b and v_c. Then we can immediately calculate i_{VS} using either the first or the last equation.

Our approach of focusing on the node voltages, defining one as ground, and then using the information provided by the voltage source has allowed us to reduce a messy problem of 5 unknowns with 4 equations to the tractable problem of 2 unknown node voltages related by two equations. The tricky issue of handling the current of the voltage source has, in essence, disappeared from view.

Taking the two equations relating v_b and v_c and working on them a bit:

$$\frac{V_S - v_b}{R_1} + \frac{v_c - v_b}{R_3} = \frac{v_b}{R_2} \qquad \frac{V_S - v_c}{R_4} + I_S = \frac{v_c - v_b}{R_3}$$

$$V_S - v_b + \frac{R_1}{R_3}\left(v_c - v_b\right) = \frac{R_1}{R_2}v_b \qquad V_S - v_c + R_4 I_S = \frac{R_4}{R_3}\left(v_c - v_b\right)$$

$$\left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right)v_b - \frac{R_1}{R_3}v_c = V_S \qquad -\frac{R_4}{R_3}v_b + \left(1 + \frac{R_4}{R_3}\right)v_c = V_S + R_4 I_S$$

Insert numbers:

$$\left(1 + \frac{40\,\Omega}{20\,\Omega} + \frac{40\,\Omega}{40\,\Omega}\right)v_b - \frac{40\,\Omega}{40\,\Omega}v_c = 8\text{ V}$$

$$-\frac{20\,\Omega}{40\,\Omega}v_b + \left(1 + \frac{20\,\Omega}{40\,\Omega}\right)v_c = 8\text{ V} + (20\,\Omega)\left(0.375\text{ mA}\right)$$

$$5v_b - v_c = 8\text{ V}$$

$$-0.5v_b + 1.5v_c = 15.5\text{ V}$$

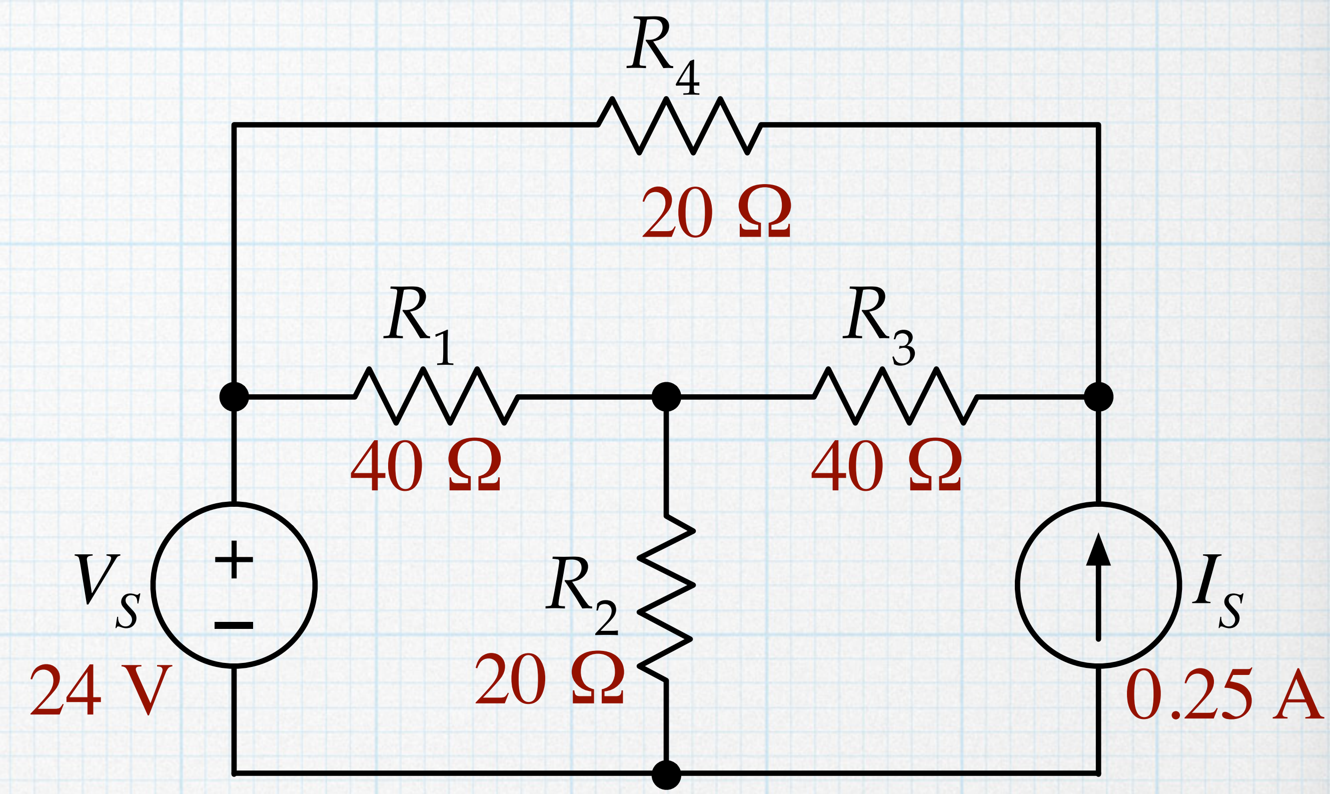
Solve the 2x2 to give: v_b = 5 V and v_c = 12 V.

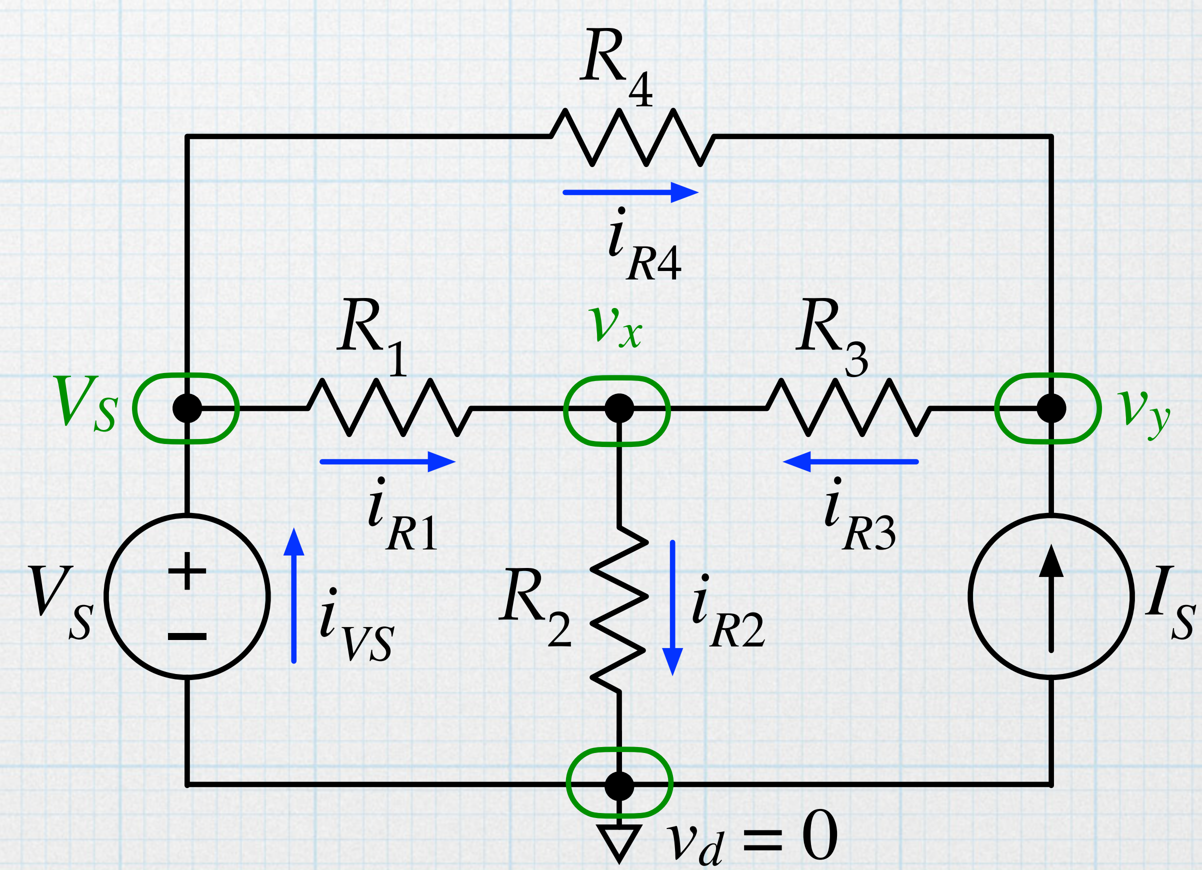
Let's step through the same problem again, but now use the knowledge gained from the first time through.

First, identify the nodes and choose one to be ground. Again, any node could be ground. Choose the bottom one again.

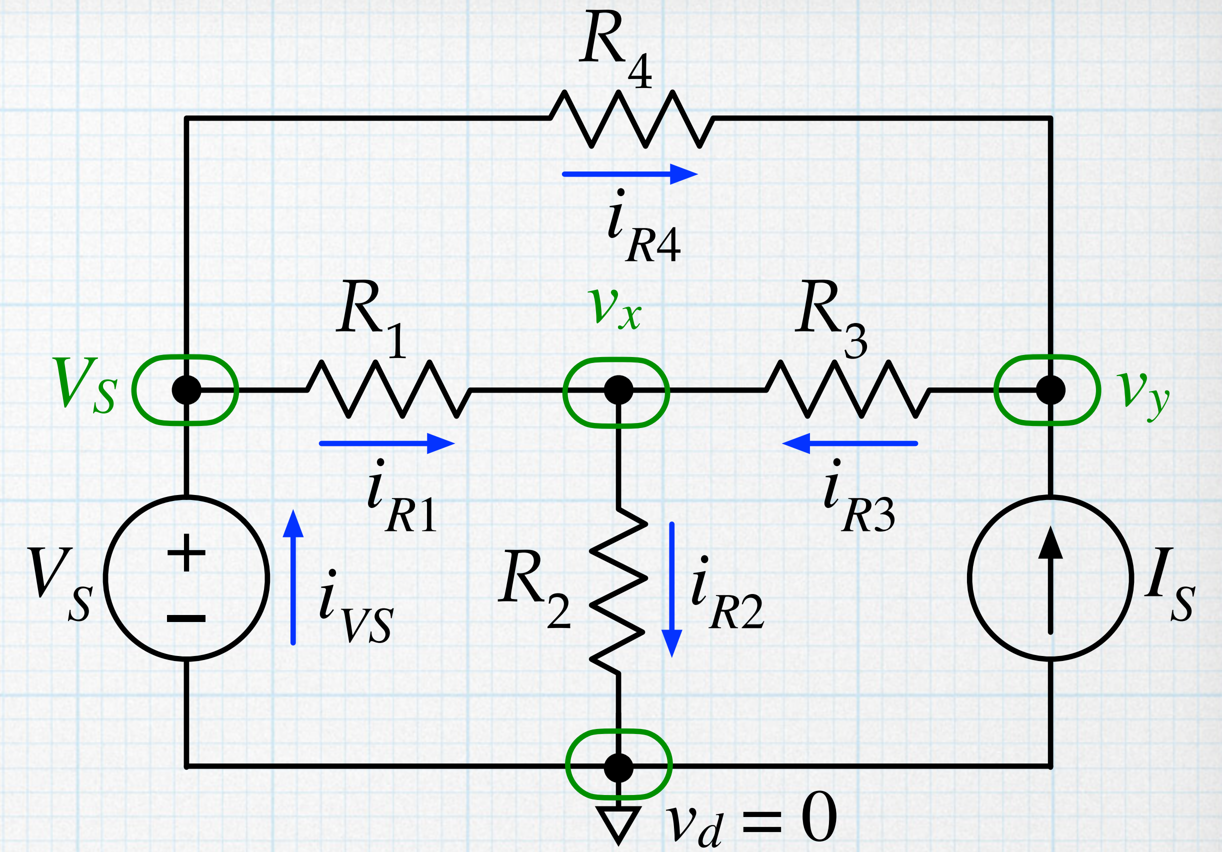
Now, with the ground chosen, we note that the voltage at the node above the voltage source must be V_S. That leaves two nodes where the voltage is not known. This time, we will label the voltages v_x and v_y. (The specific names are irrelevant.)

Identify the currents flowing into and out of the unknown nodes. Since we don't yet know the currents, we can choose the directions however we want.





The currents that enter and leave the two unknown nodes are i_{R1}, i_{R2}, i_{R3}, i_{R4}, and I_S. Note that the troublesome i_{VS} won't be involved, since it does flow into or out of of an unknown node. This is good!



At each of the unknown nodes, use KCL to balance the currents flowing in and out:

x. $i_{R1} + i_{R3} = i_{R2}$

y. $i_{R4} + I_S = i_{R3}$

Next, use Ohm's law to write each resistor current in terms of the node voltages on either end of the resistor: $i_{R1} = \dfrac{V_S - v_x}{R_1}$, etc.

$$\frac{V_S - v_x}{R_1} + \frac{v_y - v_x}{R_3} = \frac{v_x}{R_2} \qquad \frac{V_S - v_y}{R_4} + I_S = \frac{v_y - v_x}{R_3}$$

These are the two node-voltage equations that can be solved to find the two unknown node voltages. The rest is just math.

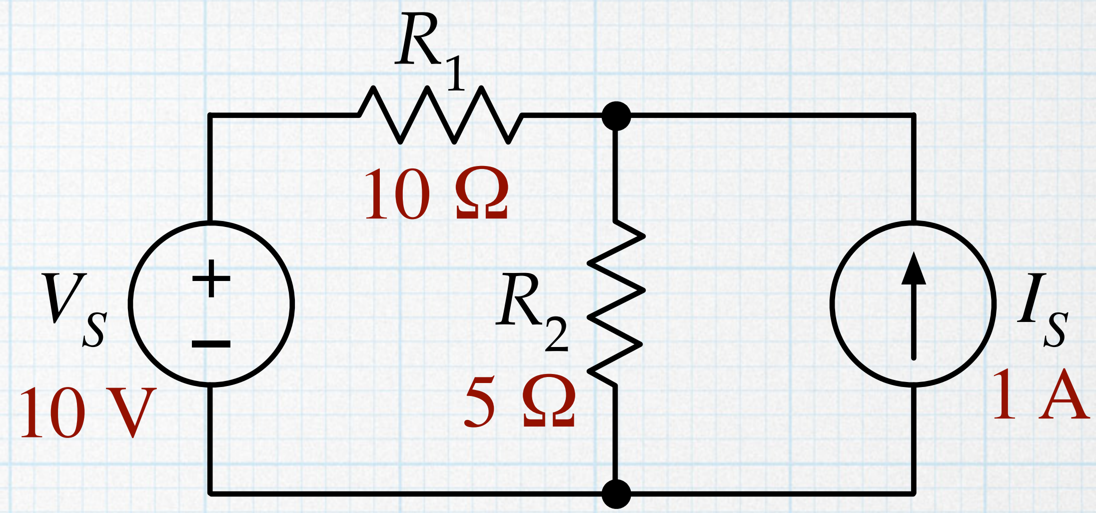
This approach is known as the *node-voltage method*.

The node-voltage method

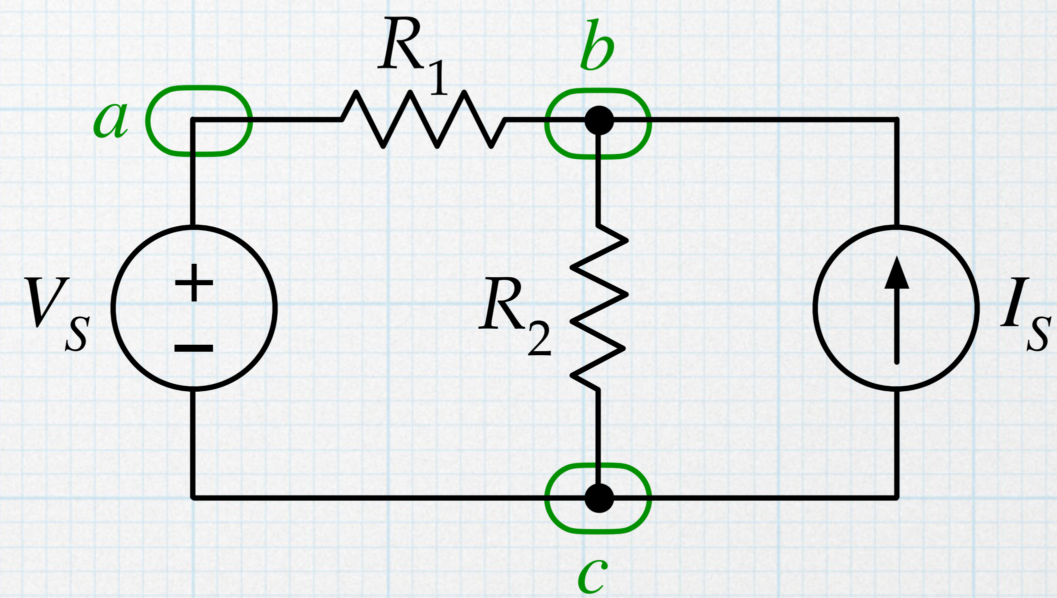
1. Identify all of the nodes in the circuit.
2. Choose one node to be ground. In principle, the choice is arbitrary, but, if possible, choose a node that is connected to a voltage source. The chosen node is assigned a voltage of 0.
3. Identify nodes for which the voltage is known due to sources.
4. If possible, use short cuts to eliminate any non-essential nodes.
5. Assign variables for the voltages at the remaining unknown nodes.
6. Assign currents to all of the branches connected to the nodes. In principle, the direction is arbitrary. Label the voltage polarity for any resistors. (Make sure that that voltage polarities match the current direction!)
7. Write KCL equations relating the currents at each of the unknown nodes.
8. Use Ohm's law to express resistor currents in terms of the (unknown) node voltages on either side of the resistor.
9. Substitute the resistor currents into the KCL equations to form the node-voltage equations — a set of equations relating the unknown node voltages.
10. Do the math to solve the equations and determine the node voltages. Determine currents, powers, etc., if needed.

Example 1

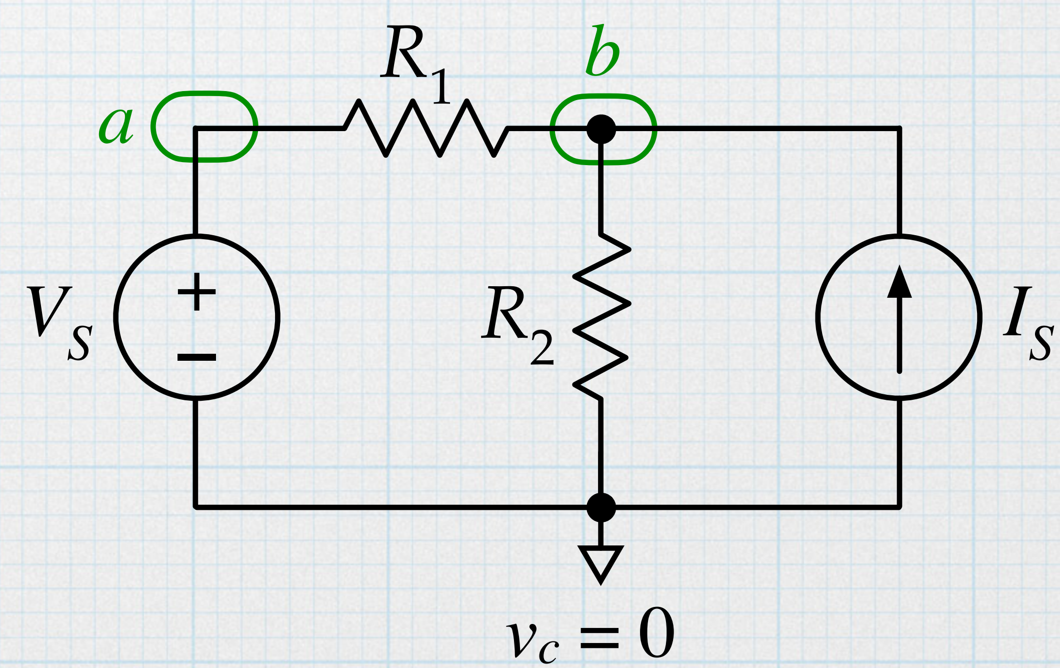
Apply the node-voltage method to the "2 source – 2 resistor" problem.



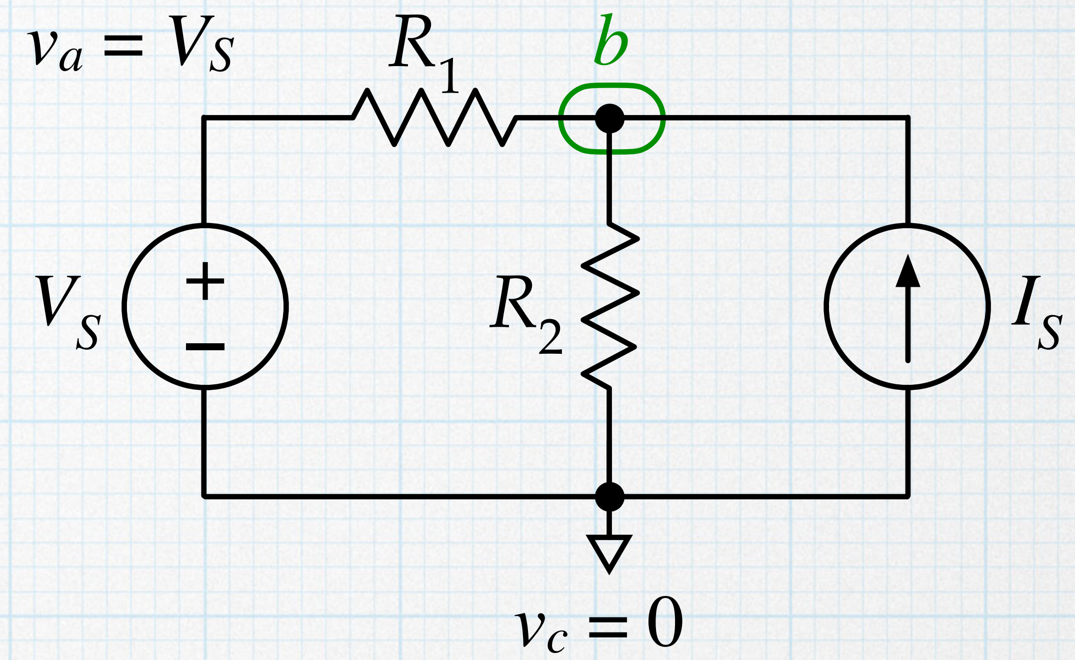
Step 1 – Identify the nodes in the circuit. Three in this case.



Step 2 – Choose one to be ground. We choose node c in this case.

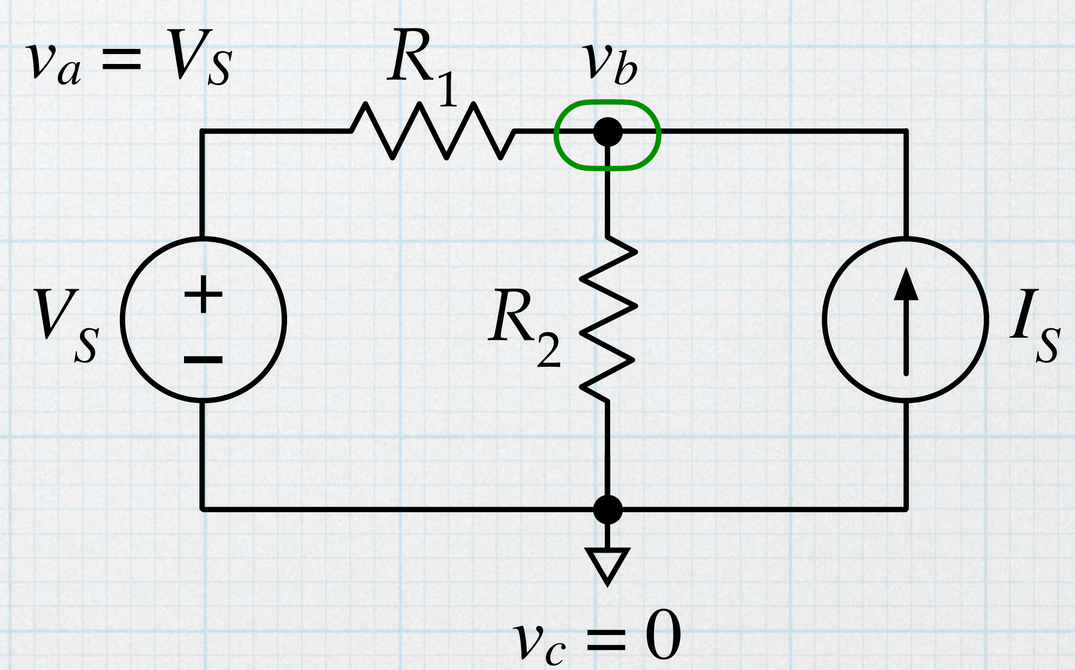


Step 3 – Identify other nodes for which the voltage is known. In this case, the source V_S between ground and node a means that $v_a = V_S$.

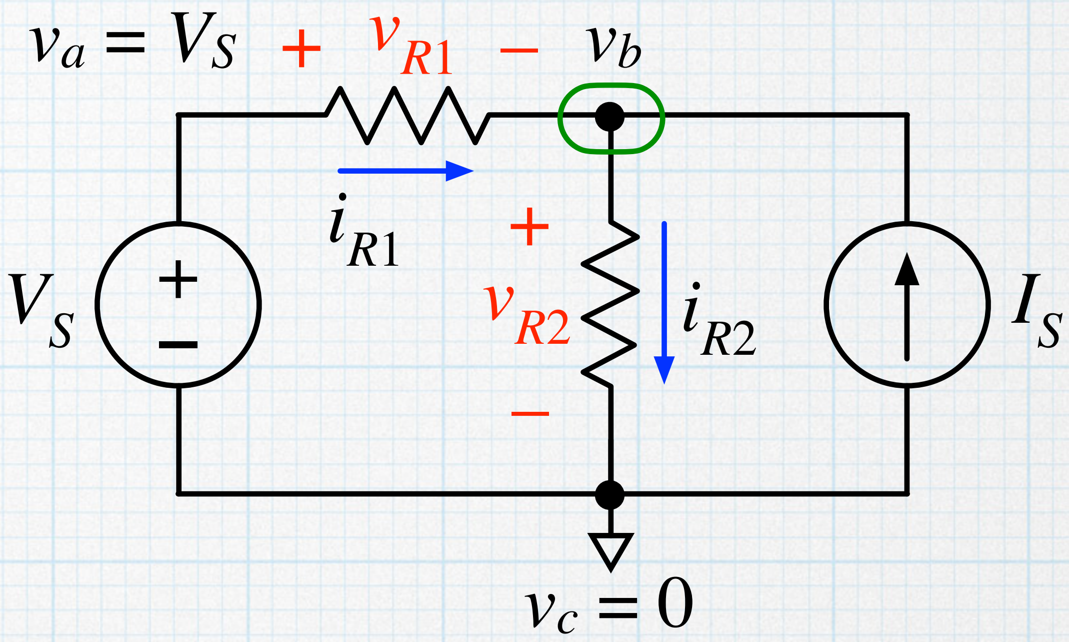


Step 4 – Reduce the circuit using series or parallel combinations. For this circuit, there are no simplifications.

Step 5 – Assign variables for the voltages at the remaining unknown nodes. In this case, there is only one unknown node voltage.



Step 6 – Assign currents to all of the branches connected to the nodes. In principle, the direction is arbitrary. Label the voltage polarity for any resistors. (Be sure to get the polarities correct!)



Step 7 – Write KCL equations relating the currents at each of the unknown nodes. In this case, there is only one equation.

$$i_{R1} + I_S = i_{R2}$$

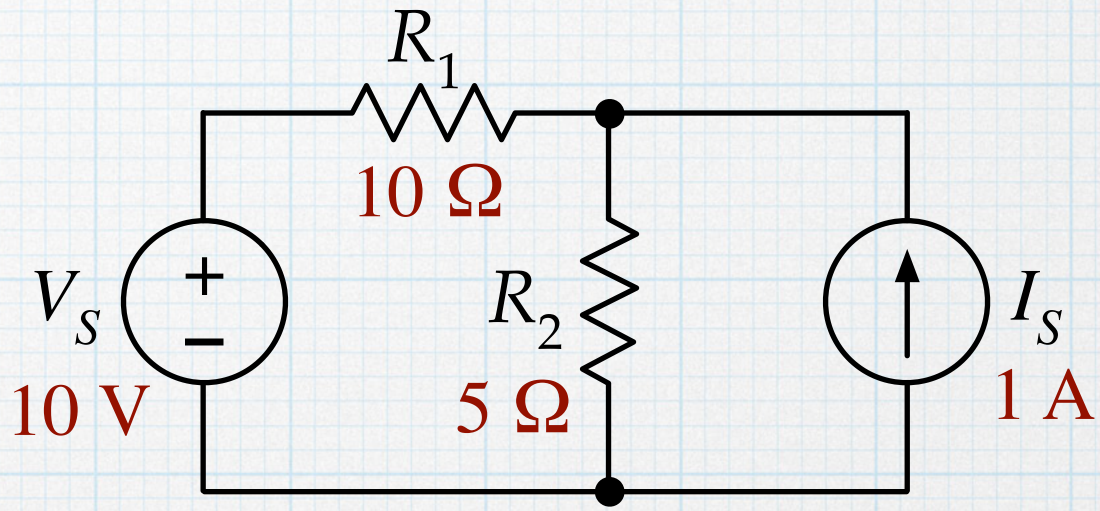
Step 8 – Use Ohm's law to express resistor currents in terms of the (unknown) node voltages on either side of the resistor.

$$i_{R1} = \frac{v_{R1}}{R_1} = \frac{v_a - v_b}{R_1} = \frac{V_S - v_b}{R_1}$$

$$i_{R2} = \frac{v_{R2}}{R_2} = \frac{v_b - v_c}{R_2} = \frac{v_b}{R_2}$$

Step 9 – Substitute the resistor currents into the KCL equation to form the node-voltage equations — a set of equations relating the unknown node voltages.

$$\frac{V_S - v_b}{R_1} + I_S = \frac{v_b}{R_2}$$



Step 10 – Do the math to find the node voltage.

$$\frac{V_S - v_b}{R_1} + I_S = \frac{v_b}{R_2}$$

$$V_S - v_b + I_S R_1 = \frac{R_1}{R_2} v_b$$

$$\left(1 + \frac{R_1}{R_2}\right) v_b = V_S + I_S R_1$$

$$v_b = \frac{V_S + I_S R_1}{1 + \frac{R_1}{R_2}}$$

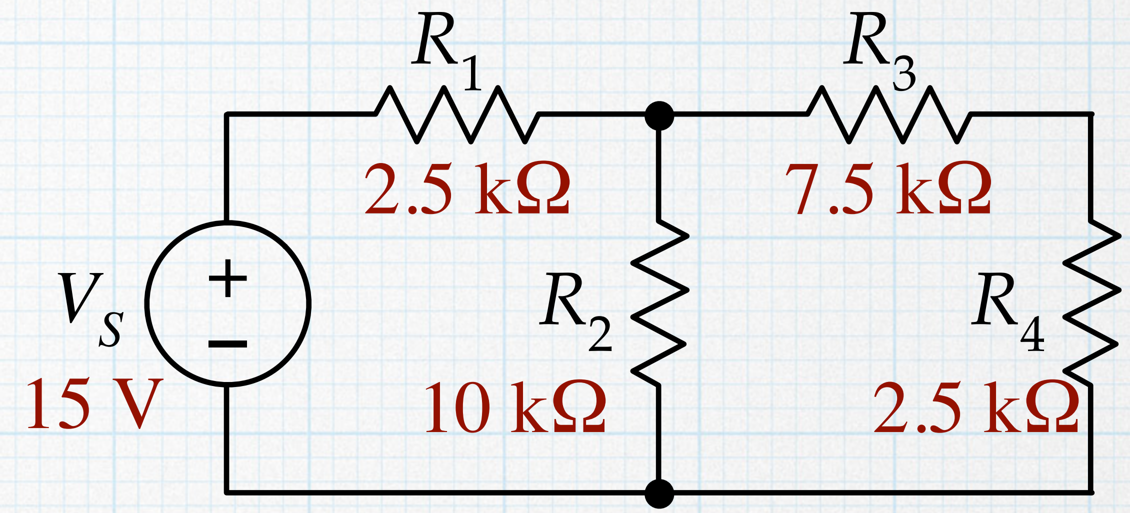
$$v_b = \frac{10\,\text{V} + (1\,\text{A})(10\,\Omega)}{1 + \frac{10\,\Omega}{5\,\Omega}}$$

$$v_b = 6.67\,\text{V}$$

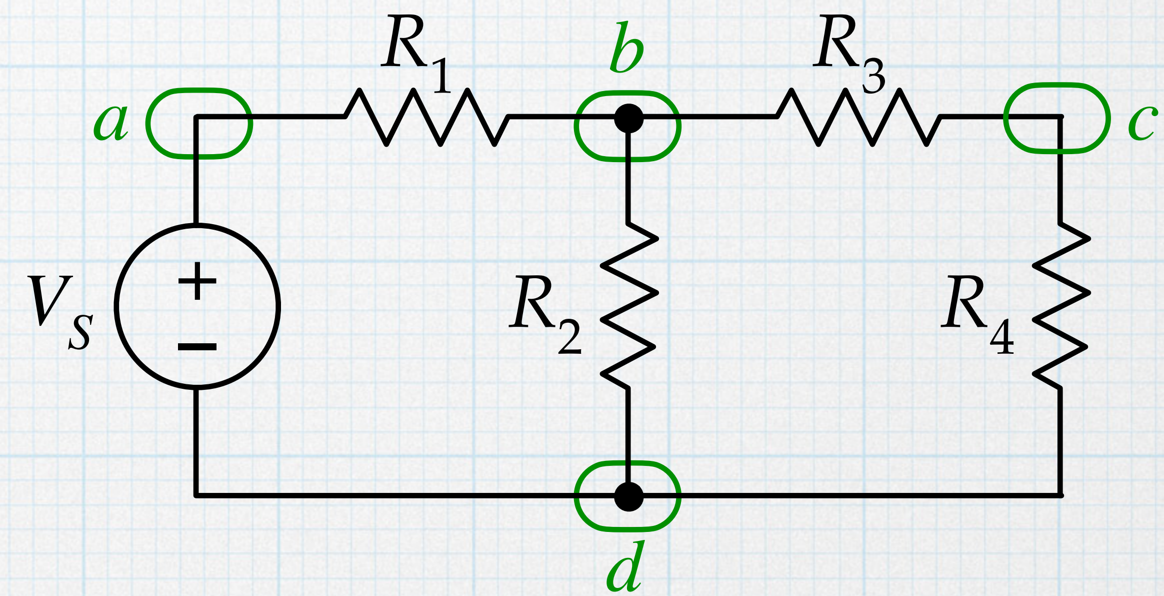
Once v_b is known, the currents and powers are easily found using Kirchoff's laws.

Example 2

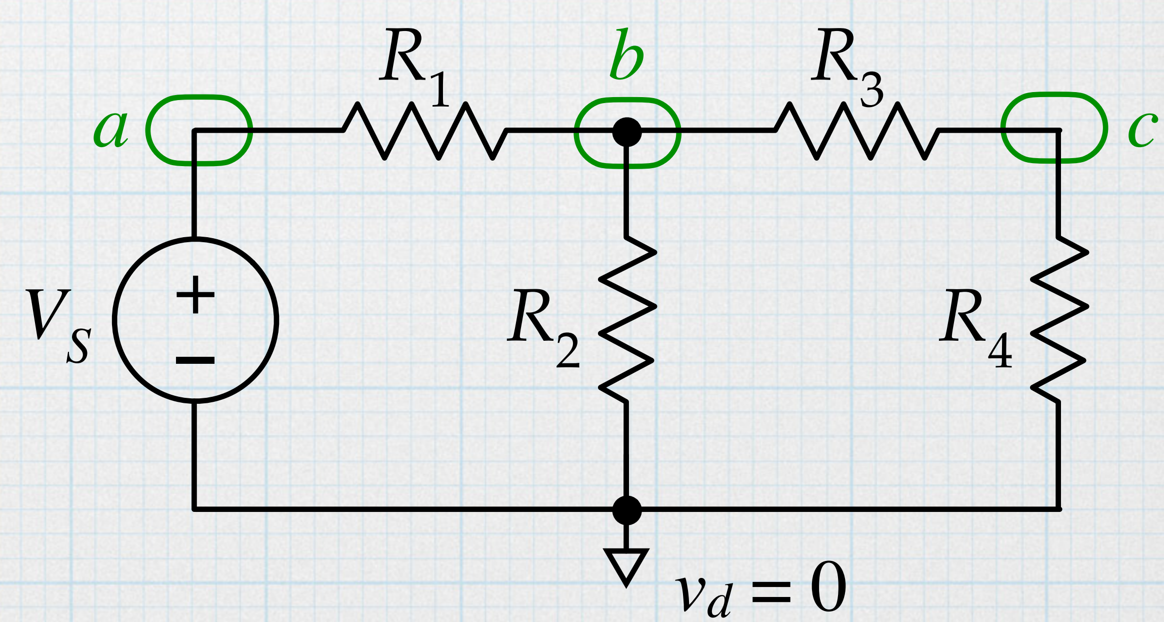
Let's apply the node voltage method to the simple ladder circuit shown. Recall that we solved this circuit earlier by using the voltage divider method twice.



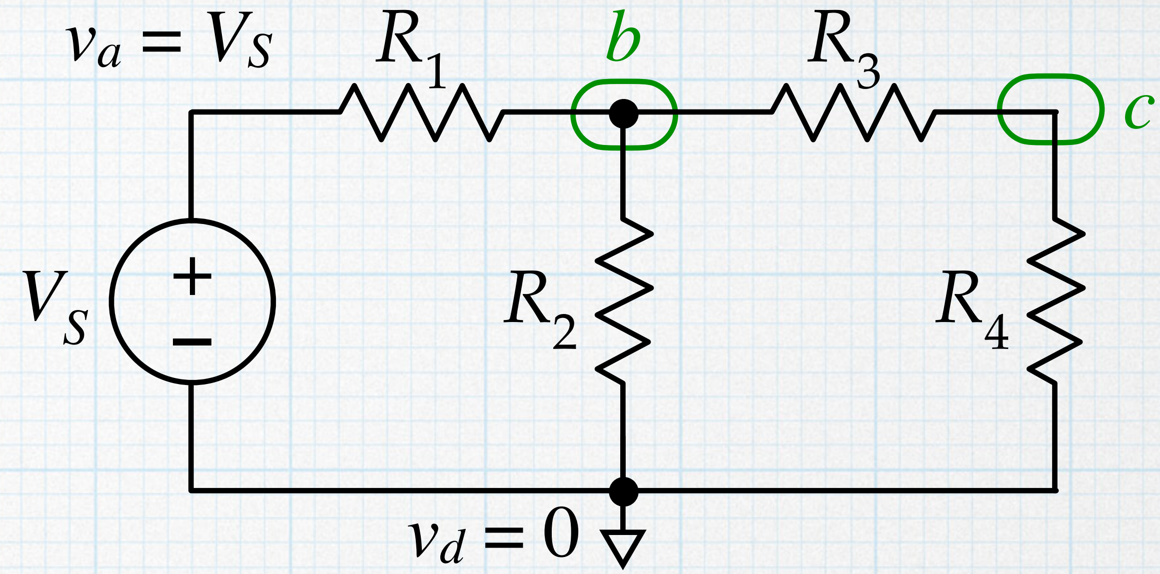
Step 1 – Identify the nodes in the circuit. Four in this case.



Step 2 – Choose one to be ground. Nodes *a* or *d* would be good choices — we will go with *d*.

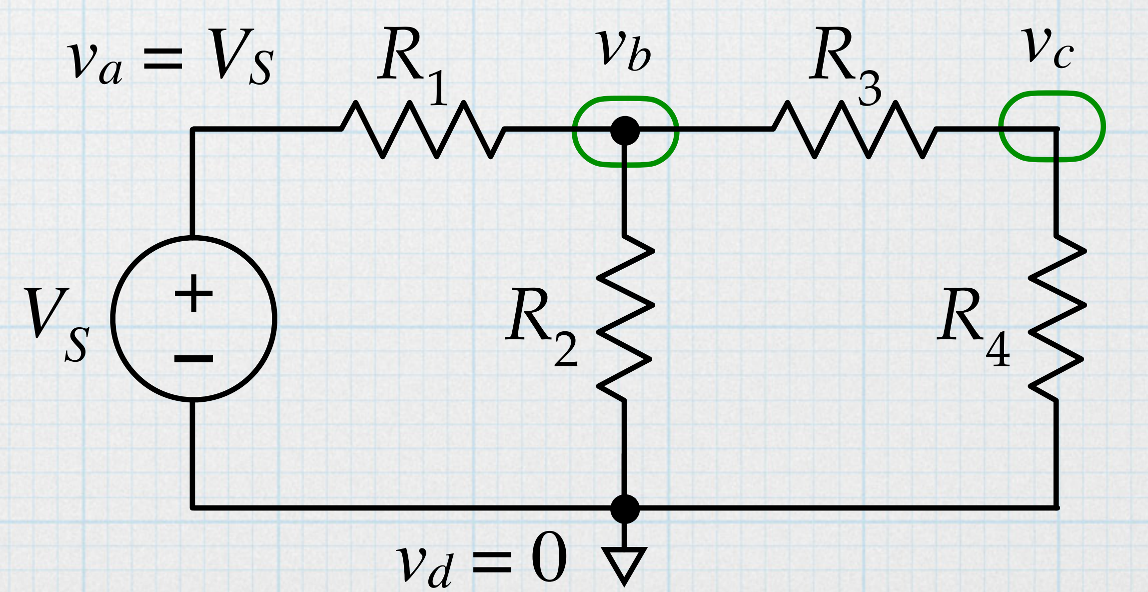


Step 3 – Identify other nodes for which the voltage is known. As in the previous example, the voltage source causes node a to have voltage $v_a = V_S$.

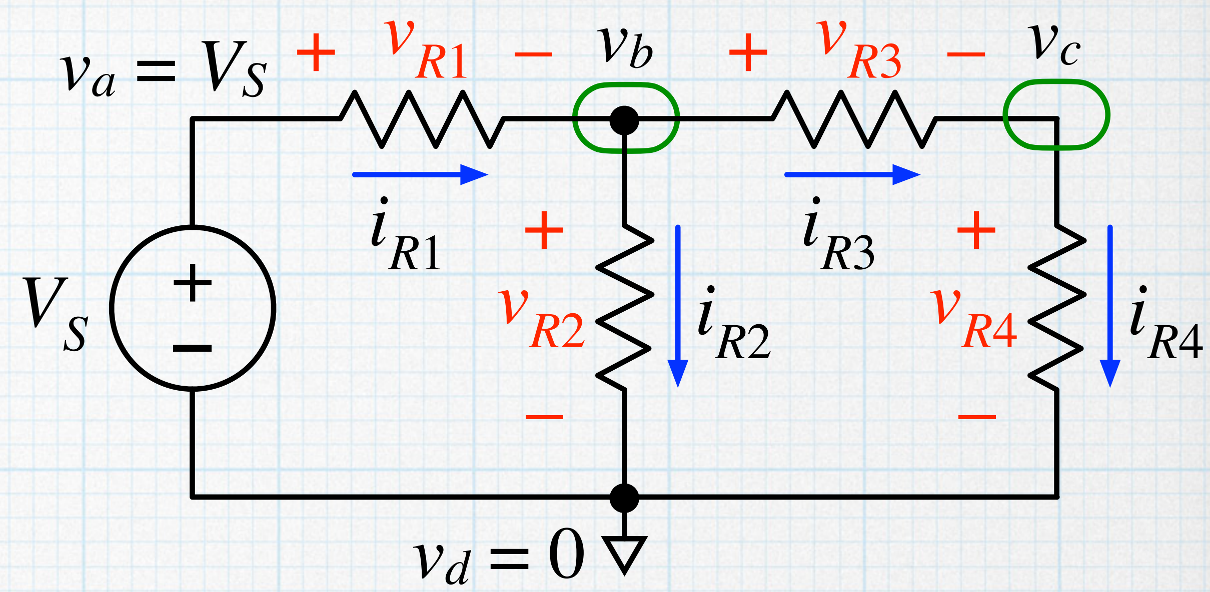


Step 4 – Reduce the circuit using series or parallel combinations. In this case, we could eliminate node c by combining R_3 and R_4 and treating them as a single resistor. Then the problem reduces to having a single unknown (v_b) and it could be handled easily using a voltage divider, taking us back to our earlier method. However, to better illustrate the node-voltage method, we will keep the two resistors separate, with node c between them.

Step 5 – Assign variables for the voltages at the remaining unknown nodes. In this case, there are two unknown node voltages.



Step 6 – Assign currents to all of the branches connected to the nodes. In principle, the direction is arbitrary. Label the voltage polarity for any resistors — take care to match voltage polarity to current direction.



Step 7 – Write KCL equations relating the currents at each of the unknown nodes.

$$i_{R1} = i_{R2} + i_{R3}$$

$$i_{R3} = i_{R4}$$

Step 8 – Use Ohm's law to express resistor currents in terms of the (unknown) node voltages on either side of the resistor.

$$i_{R1} = \frac{v_{R1}}{R_1} = \frac{v_a - v_b}{R_1} = \frac{V_S - v_b}{R_1}$$

$$i_{R2} = \frac{v_{R2}}{R_2} = \frac{v_b - v_d}{R_2} = \frac{v_b}{R_2}$$

$$i_{R3} = \frac{v_{R3}}{R_3} = \frac{v_b - v_c}{R_3}$$

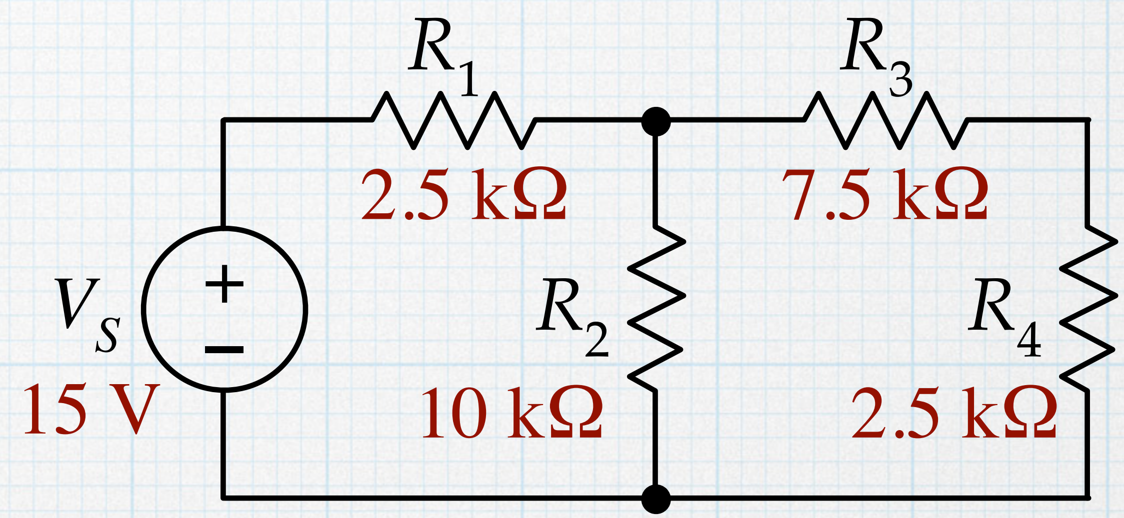
$$i_{R4} = \frac{v_{R4}}{R_4} = \frac{v_c - v_d}{R_4} = \frac{v_4}{R_4}$$

Step 9 – Substitute the resistor currents into the KCL equation(s) to form the node-voltage equations.

$$\frac{V_S - v_b}{R_1} = \frac{v_b}{R_2} + \frac{v_b - v_c}{R_3}$$

$$\frac{v_b - v_c}{R_3} = \frac{v_c}{R_4}$$

Step 10 – Do the math to find the node voltages.



$$\begin{cases} V_S - v_b = \dfrac{R_1}{R_2} v_b + \dfrac{R_1}{R_3}\left(v_b - v_c\right) \\ v_b - v_c = \dfrac{R_3}{R_4} v_c \end{cases}$$

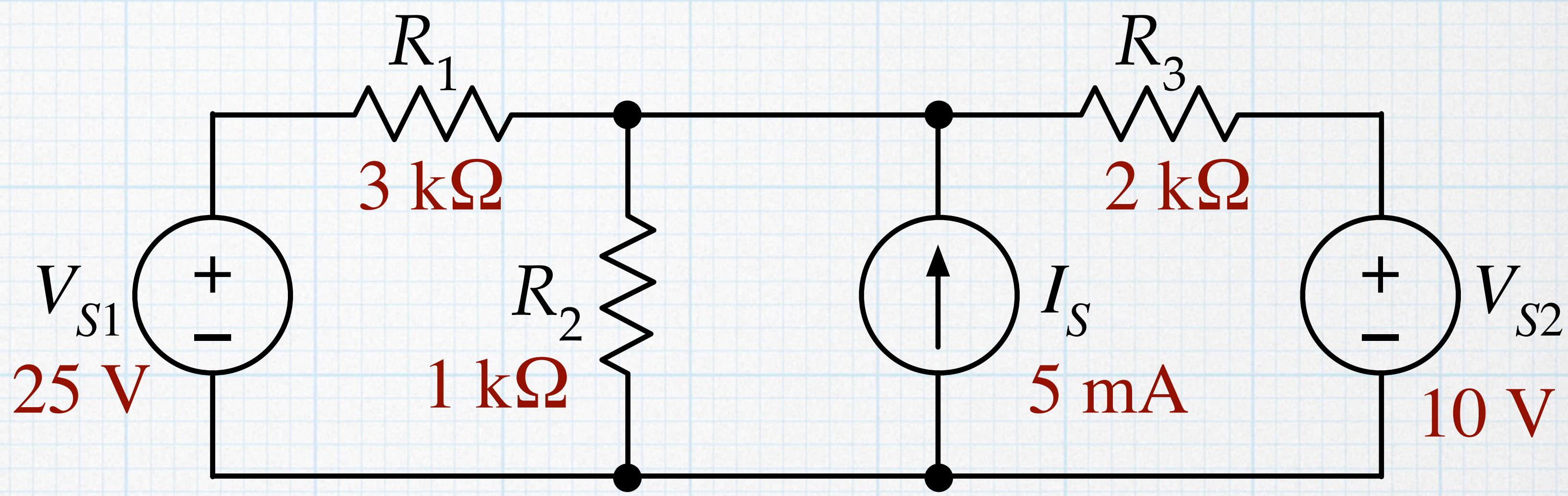
$$\begin{cases} \left(1 + \dfrac{R_1}{R_2} + \dfrac{R_1}{R_3}\right) v_b - \dfrac{R_1}{R_3} v_c = V_s \\ v_b - \left(1 + \dfrac{R_3}{R_4}\right) v_c = 0 \end{cases}$$

$$\begin{cases} 1.5833 v_b - 0.333 v_c = 15\text{ V} \\ v_b - 4 v_c = 0 \end{cases}$$

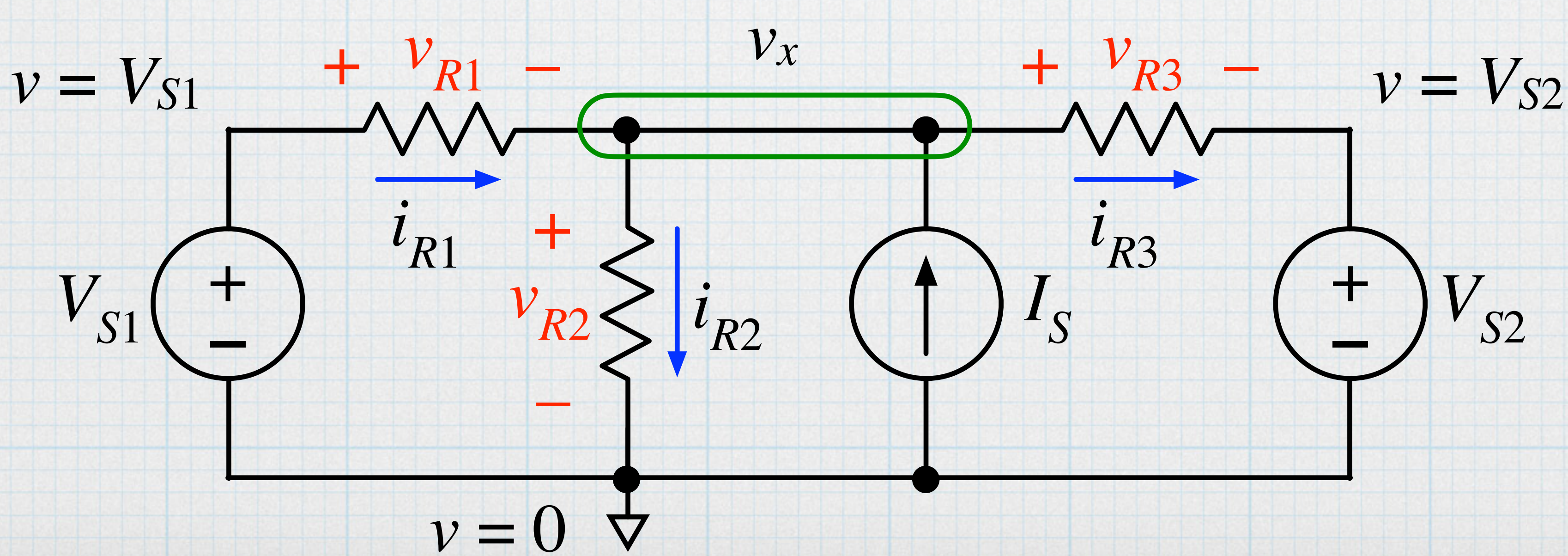
Two equations, two unknowns:
Solve to give: $v_b = 10$ V, $v_c = 2.5$ V

Example 3

As we become more familiar with the node-voltage procedure, we can probably do some of the steps by "inspection", without writing out everything.



We see that there are four nodes in the circuit. Making things easier, there are two voltage sources that share a common connection. It makes sense to choose that node as ground. The ground and the two voltage sources mean that we already know the voltages of three of the nodes. This becomes a "one-node" circuit and should be easy to solve.



There are no series/parallel simplifications, so we can jump directly to writing the KCL equation.

$$\text{at node } x : i_{R1} + I_S = i_{R2} + i_{R3}$$

As we become better at recognizing how the node voltages relate to the resistor currents, we can immediately re-write the currents above in terms of node voltages and resistors:

$$\frac{V_{S1} - v_x}{R_1} + I_S = \frac{v_x}{R_2} + \frac{v_x - V_{S2}}{R_3}$$

The circuit analysis is done, and the rest is just math.

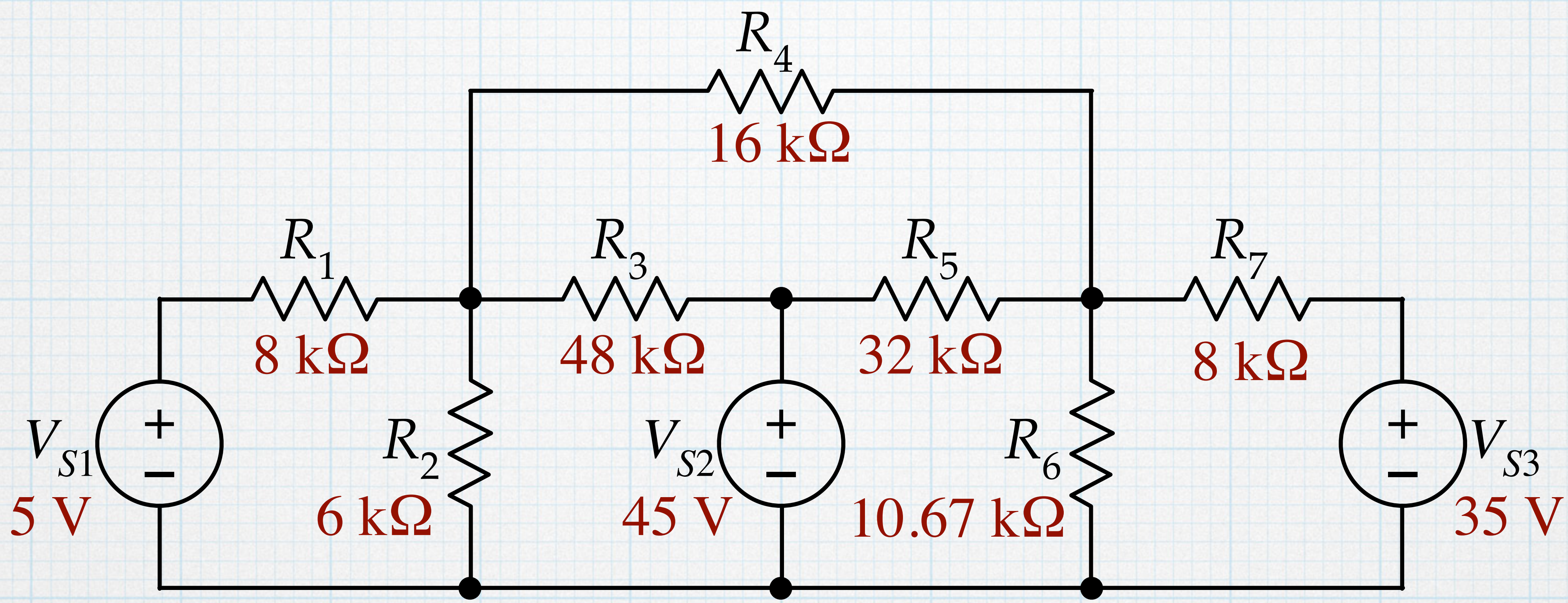
$$V_{S1} - v_x + R_1 I_S = \frac{R_1}{R_2} V_x + \frac{R_1}{R_3} v_x - \frac{R_1}{R_3} V_{S2}$$

$$\left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right) v_x = V_{S1} + R_1 I_S + \frac{R_1}{R_3} V_{S2}$$

$$v_x = \frac{V_{S1} + R_1 I_S + \frac{R_1}{R_3} V_{S2}}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}} = \frac{25\,\mathrm{V} + \left(3\,\mathrm{k\Omega}\right)\left(5\,\mathrm{mA}\right) + \frac{3\,\mathrm{k\Omega}}{2\,\mathrm{k\Omega}}\left(25\,\mathrm{V}\right)}{1 + \frac{3\,\mathrm{k\Omega}}{1\,\mathrm{k\Omega}} + \frac{3\,\mathrm{k\Omega}}{2\,\mathrm{k\Omega}}} = 10\,\mathrm{V}$$

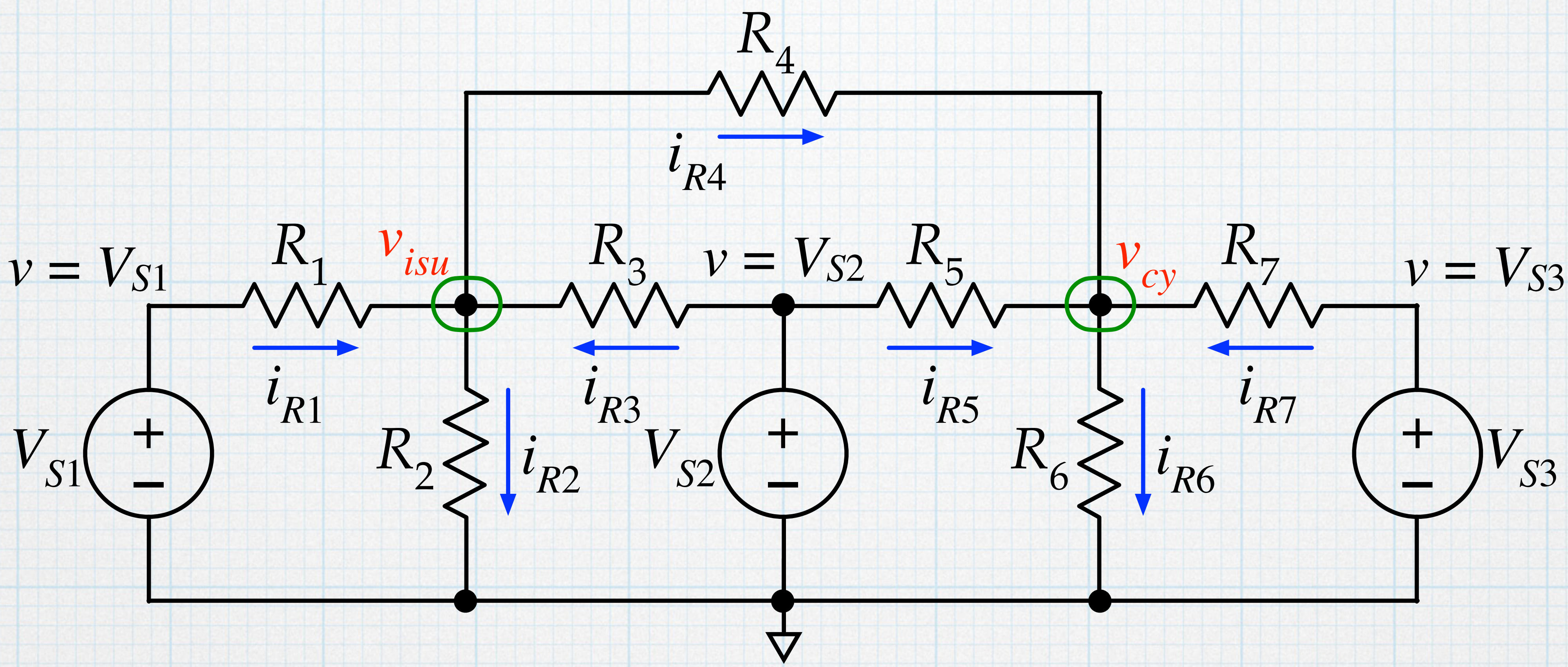
Example 4

Practice makes perfect. Here is a circuit with three voltage sources.



There are six nodes in the circuit. Fortunately, if we choose ground at the bottom once again, then three of the nodes will have known voltages due to the sources.

There are no series/parallel simplifications, so we can label the two unknown voltages and define the currents. Again, directions are arbitrary at this point.



Write the KCL equations at the two nodes:

$$\text{at node “}isu\text{” : } i_{R1} + i_{R3} = i_{R2} + i_{R4}$$

$$\text{at node “}cy\text{” : } i_{R4} + i_{R5} + i_{R7} = i_{R6}$$

Convert the KCL equations to node-voltage equations using Ohm's law and the node voltages.

$$\frac{V_{S1} - v_{isu}}{R_1} + \frac{V_{S2} - v_{isu}}{R_3} = \frac{v_{isu}}{R_2} + \frac{v_{isu} - v_{cy}}{R_4}$$

$$\frac{V_{S2} - v_{cy}}{R_5} + \frac{V_{S3} - v_{cy}}{R_7} + \frac{v_{isu} - v_{cy}}{R_4} = \frac{v_{cy}}{R_6}$$

The circuit analysis is done, and we need to finish the math to find the voltages. Start by re-arranging.

$$\left(1 + \frac{R_3}{R_1} + \frac{R_3}{R_2} + \frac{R_3}{R_4}\right) v_{isu} - \frac{R_3}{R_4} v_{cy} = V_{S2} + \frac{R_3}{R_1} V_{S1}$$

$$-\frac{R_5}{R_4} v_{isu} + \left(1 + \frac{R_5}{R_4} + \frac{R_5}{R_6} + \frac{R_5}{R_7}\right) v_{cy} = V_{S2} + \frac{R_5}{R_7} V_{S3}$$

Plug in the values:

$$\left(1+\frac{48\,\mathrm{k\Omega}}{8\,\mathrm{k\Omega}}+\frac{48\,\mathrm{k\Omega}}{6\,\mathrm{k\Omega}}+\frac{48\,\mathrm{k\Omega}}{16\,\mathrm{k\Omega}}\right)v_{isu}-\frac{48\,\mathrm{k\Omega}}{16\,\mathrm{k\Omega}}v_{cy}=45\,\mathrm{V}+\frac{48\,\mathrm{k\Omega}}{8\,\mathrm{k\Omega}}\left(5\,\mathrm{V}\right)$$

$$-\frac{32\,\mathrm{k\Omega}}{16\,\mathrm{k\Omega}}v_{isu}+\left(1+\frac{32\,\mathrm{k\Omega}}{16\,\mathrm{k\Omega}}+\frac{32\,\mathrm{k\Omega}}{10.67\,\mathrm{k\Omega}}+\frac{32\,\mathrm{k\Omega}}{8\,\mathrm{k\Omega}}\right)v_{cy}=45\,\mathrm{V}+\frac{32\,\mathrm{k\Omega}}{8\,\mathrm{k\Omega}}\left(35\,\mathrm{V}\right)$$

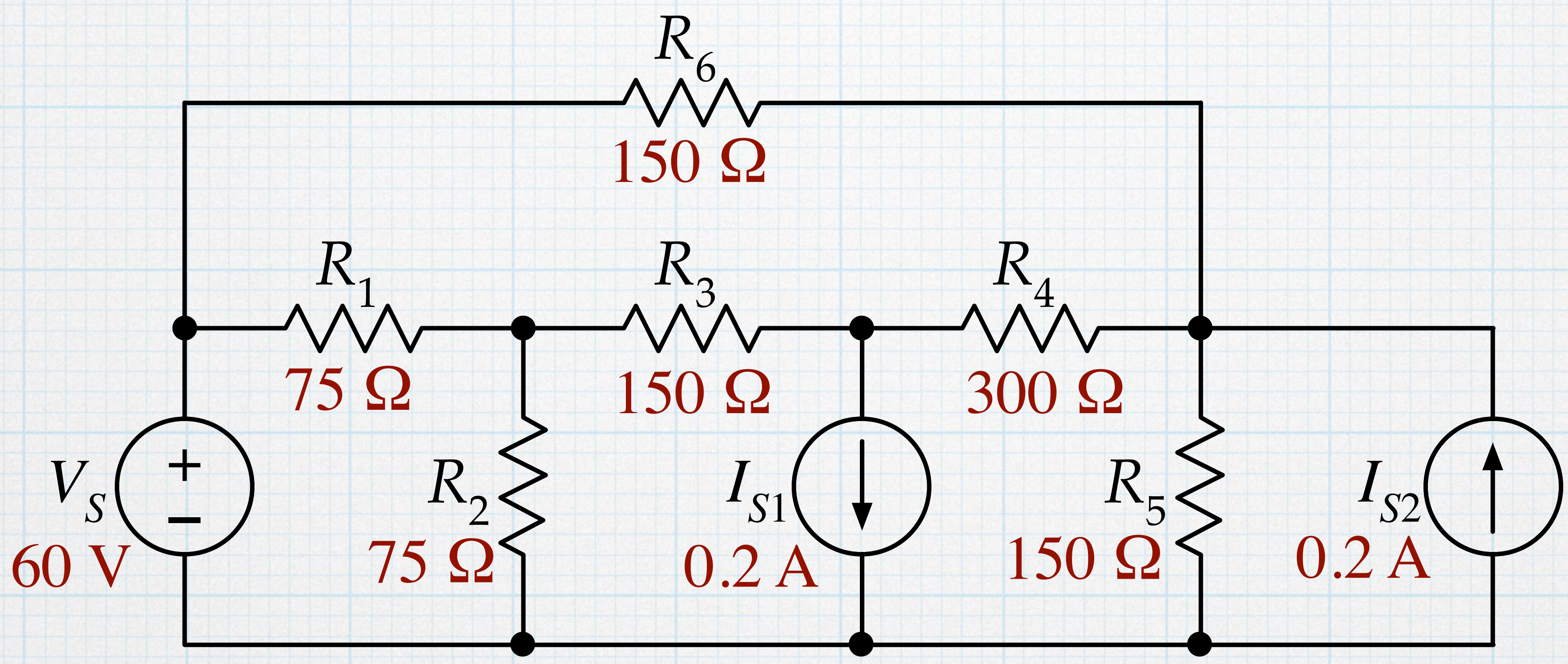
$$18v_{isu}-3v_{cy}=75\,\mathrm{V}$$

$$-2v_{isu}-10v_{cy}=185\,\mathrm{V}$$

Solve to give: $v_{isu}=7.5\,\mathrm{V}$ and $v_{cy}=20\,\mathrm{V}$

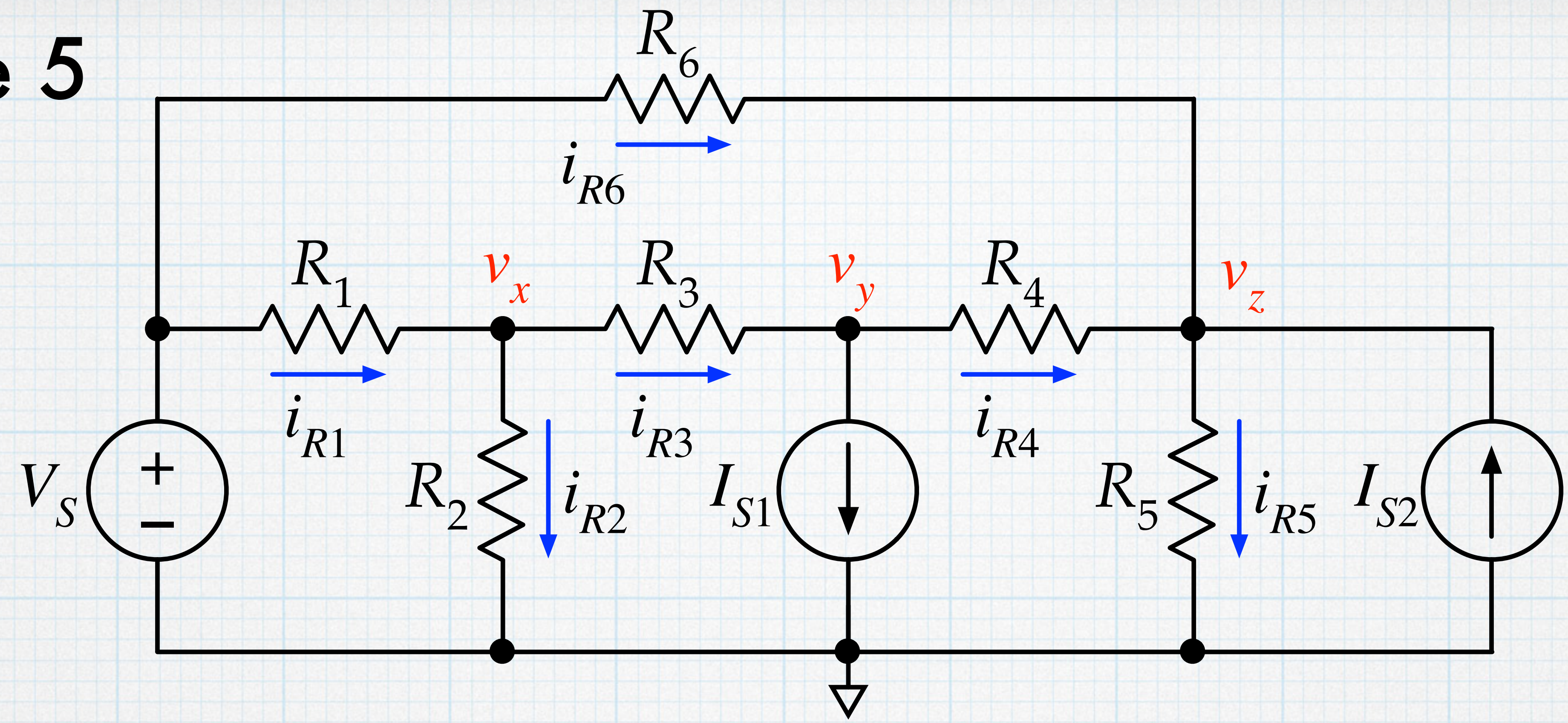
Example 5

Here is a slightly bigger circuit to tackle. The approach doesn't change.



We see that there are five nodes. Choose the ground connection — the nodes on either side of the voltage source are good options. Choosing the bottom node as ground makes the voltage at the left-hand node equal to V_S.

Example 5



That leaves three unknown node voltages. Those are labeled as above, along with currents in each of the branches connected by nodes. Write KCL equations for the currents at each of the nodes and then convert them to node-voltage equations.

$$x : i_{R1} = i_{R2} + i_{R3}$$

$$y : i_{R3} = I_{S1} + i_{R4}$$

$$z : i_{R4} + I_{R6} + I_{S2} = i_{R5}$$

$$x : \frac{V_S - v_x}{R_1} = \frac{v_x}{R_2} + \frac{v_x - v_y}{R_3}$$

$$y : \frac{v_x - v_y}{R_3} = I_S + \frac{v_y - v_z}{R_4}$$

$$z : \frac{v_y - v_z}{R_4} + \frac{V_S - v_z}{R_6} + I_{S2} = \frac{v_z}{R_5}$$

Example 5

Now take care of the math. First, re-arrange the equations:

$$x: \left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right) v_x - \frac{R_1}{R_3} v_y = V_S$$

$$y: v_x - \left(1 + \frac{R_3}{R_4}\right) v_y + \frac{R_3}{R_4} v_z = R_3 I_{S1}$$

$$z: -\frac{R_6}{R_4} v_y + \left(1 + \frac{R_6}{R_4} + \frac{R_6}{R_5}\right) v_z = V_S + R_6 I_{S2}$$

Then insert the values

$$x: 2.5 v_x - 0.5 v_y = 60\ \mathrm{V}$$

$$y: v_x - 1.5 v_y + 0.5 v_z = 30\ \mathrm{V}$$

$$z: -0.5 v_y + 2.5 v_z = 90\ \mathrm{V}$$

And then solve*

$v_x = 26$ V, $v_y = 10$ V, and $v_z = 38$ V.

There is no "correct" way to do the algebra after the NV equations have been determined. All of the examples shown in these notes use an approach that forms dimensionless resistor ratios for the coefficients. But there are other algebraic methods that are equally good.

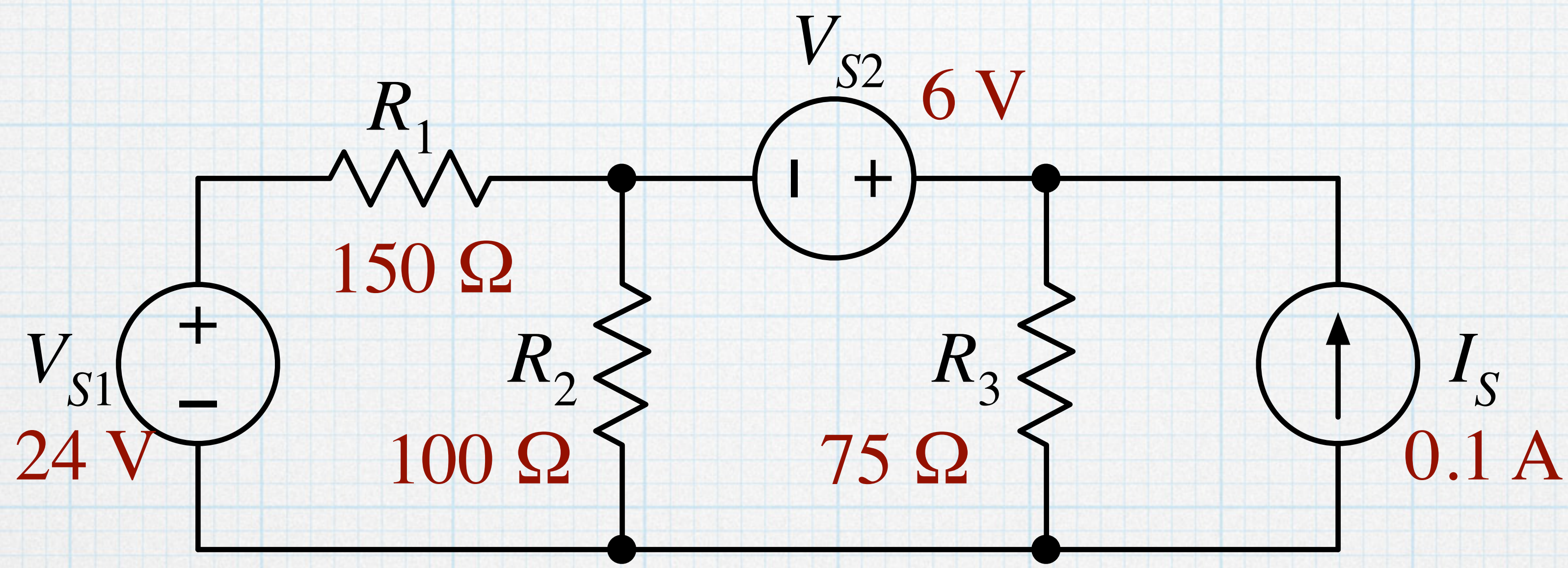
* Use an on-line calculator

- http://math.bd.psu.edu/~jpp4/finitemath/3x3solver.html
- http://www.1728.org/unknwn3.htm
- https://www.wolframalpha.com

Or use the solver on your calculator.

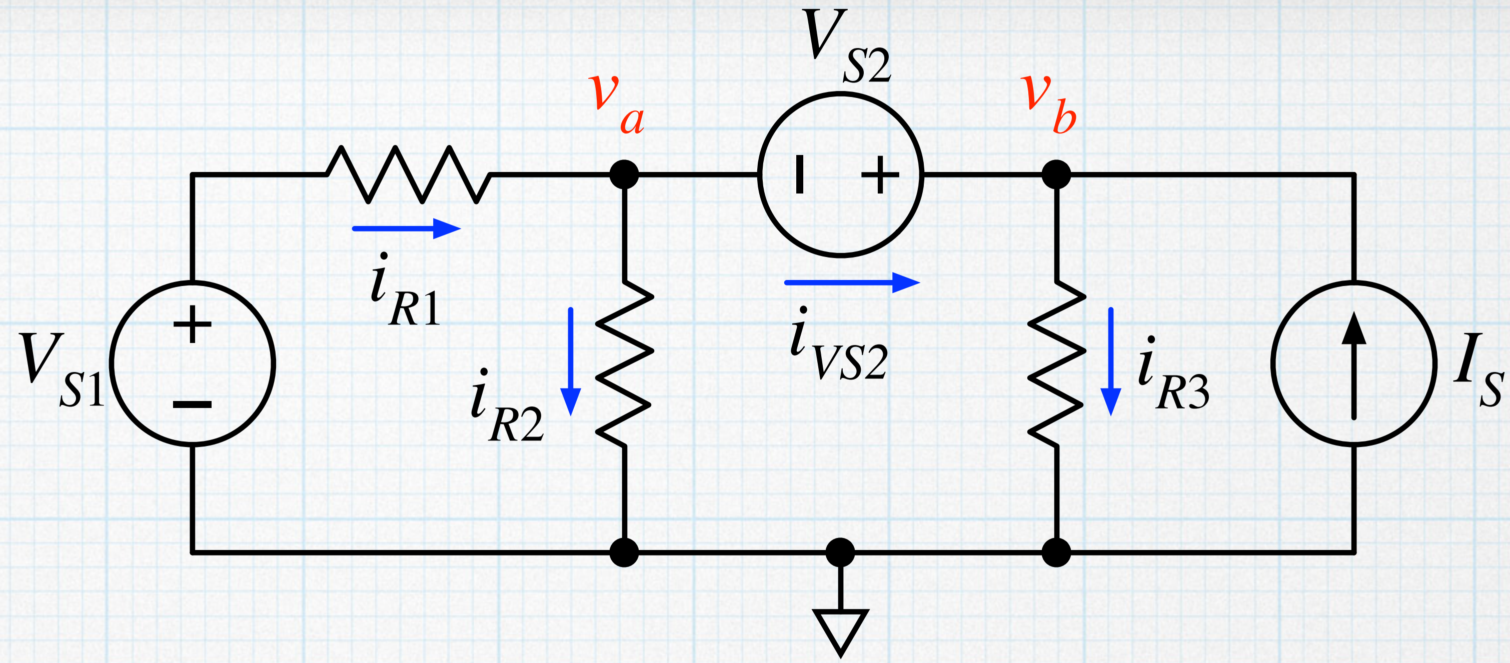
Rogue voltage sources

As described above and demonstrated in the examples, the node-voltage method works well in most circuits. However, there is one situation where the basic algorithm fails, and we must improvise a bit. Consider the circuit below.



There are four nodes, and at first glance it looks like another routine application of the method. The complication arises in choosing which node to be ground. There are two voltage sources, and they do not share a node. We need to pick a ground, and we will stick with past habit and put it at the bottom node.

There are two unknown node voltages. That seems OK.



But things get sticky when we write the two KCL equations,

$$a : i_{R1} = i_{R2} + i_{VS2}$$

$$b : i_{VS2} + I_S = I_{R3}$$

and try to turn them into node-voltage equations,

$$a : \frac{V_{S1} - v_a}{R_1} = \frac{v_a}{R_2} + i_{VS2}$$

$$b : i_{VS2} + I_S = \frac{v_b}{R_3}$$

The two expected unknowns, v_a and v_b, are joined by a third, i_{VS2} — the current through the second voltage source. Two equations, three unknowns — not good.

$$a : \frac{V_{S1} - v_a}{R_1} = \frac{v_a}{R_2} + i_{VS2}$$

$$b : i_{VS2} + I_S = \frac{v_b}{R_3}$$

We could use the second equation to write i_{VS2} in terms of I_S and v_b and then insert the result into the first equation, thus eliminating i_{VS2}. But the result is one equation in two unknowns. So that is not helpful.

The correct approach is to find another equation relating the quantities that can be added to the mix. We can call the extra relation the *auxiliary equation*. In this case, V_{S2} — with its unknown current — is causing the difficulty in analyzing this circuit, but it also offers the way out of the conundrum. From the definition of a voltage source, we can write the auxiliary equation: $V_{S2} = v_b - v_a$. Now, we have three equations in three unknowns, and the path to a solution is clear.

$$a : \frac{V_{S1} - v_a}{R_1} = \frac{v_a}{R_2} + i_{VS2}$$

$$b : i_{VS2} + I_S = \frac{v_b}{R_3}$$

$$\text{aux} : V_{S2} = v_b - v_a$$

There are several ways to handle the math. One is to eliminate i_{VS2} from a and b, as suggested above.

$$\frac{V_{S1} - v_a}{R_1} = \frac{v_a}{R_2} + \frac{v_b}{R_3} - I_S$$

$$V_{S2} = v_b - v_a$$

Then use the auxiliary equation to solve for v_b and substitute into the other equation.

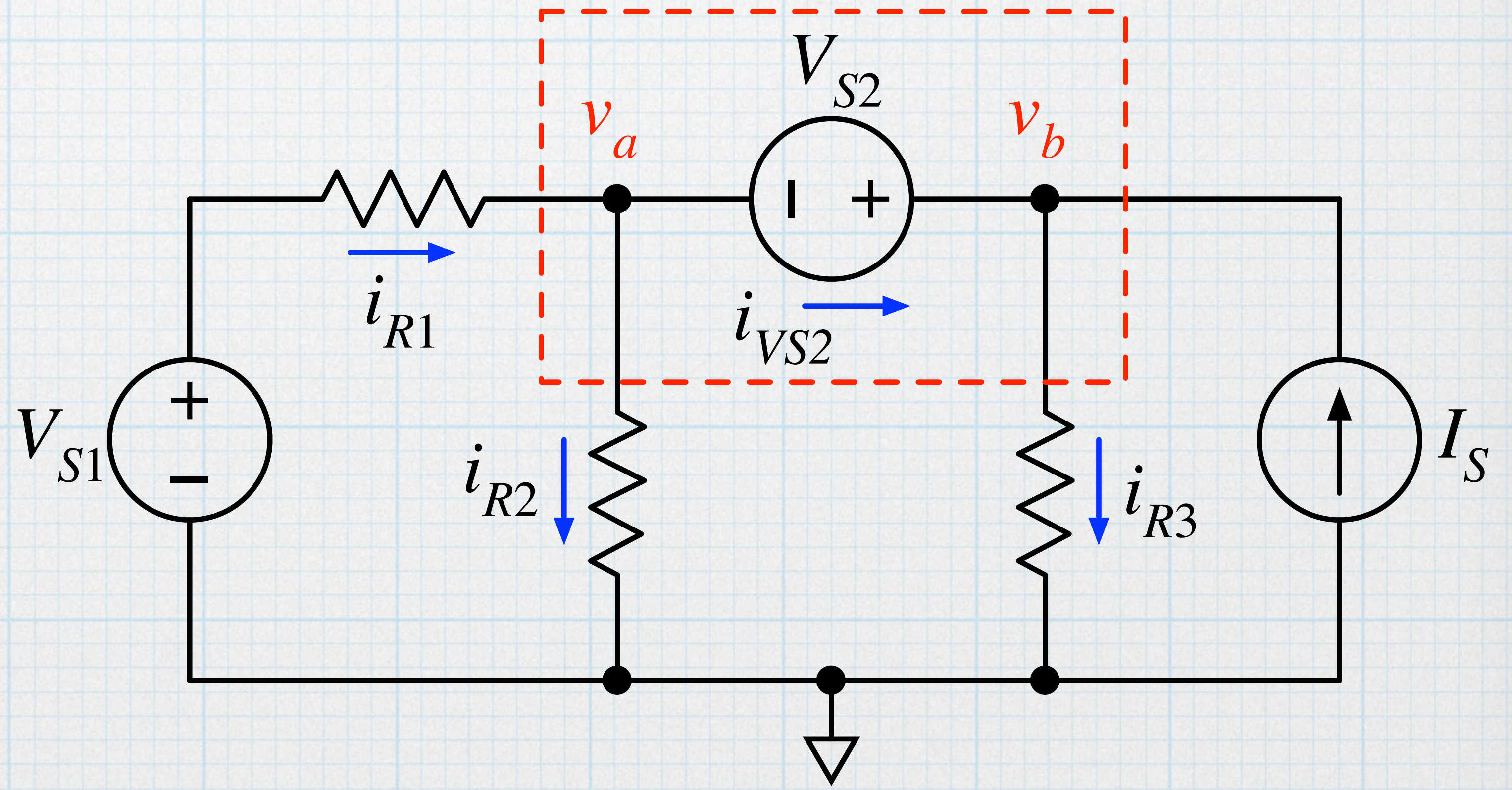
$$\frac{V_{S1} - v_a}{R_1} = \frac{v_a}{R_2} + \frac{V_{S2} + v_a}{R_3} - I_S$$

Solve for v_a: $v_a = \dfrac{V_{S1} + R_1 I_S - \frac{R_1}{R_3} \cdot V_{S2}}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}} = 6\text{ V}$ and $v_b = v_a + V_{S2} = 12$ V.

The super node

Some textbooks offer a slight variation on the approach just described. The alternate method avoids even having to acknowledge that the troublesome voltage source has an unknown current. The trick is to create a "really big node" or *super node* that completely encloses the offending voltage source. The method relies on the fact that KCL applies to any sized entity. It doesn't matter how big it is — what goes in must come out. We typically apply KCL to the bundle of wires that makes up a node, but it works just as well if we make a little box that contains some portion of the circuit — what goes into the box must come out.

In particular for this circuit, we make a box that contains V_{S2} and the nodes on either side of it, as shown at right. The box is the super node. Then apply KCL to the box:



$$i_{R1} + I_S = i_{R2} + i_{R3}$$

Note that i_{VS2} does not appear!

$$i_{R1} + I_S = i_{R2} + i_{R3}$$

Now that we have a KCL equation, we can forget about the super node. (The super node does not have a single voltage — on one side the voltage is v_a and the other it is v_b. So in this sense, the super node is not like "regular" nodes that we typically use. The super node's only purpose is to come up with a simpler KCL equation.)

Now we proceed as before — use Ohm's law to express the currents in terms of the voltages on either sides. Here we use v_a and v_b.

$$\frac{V_{S1} - v_a}{R_1} + I_S = \frac{v_a}{R_2} + \frac{v_b}{R_3}$$

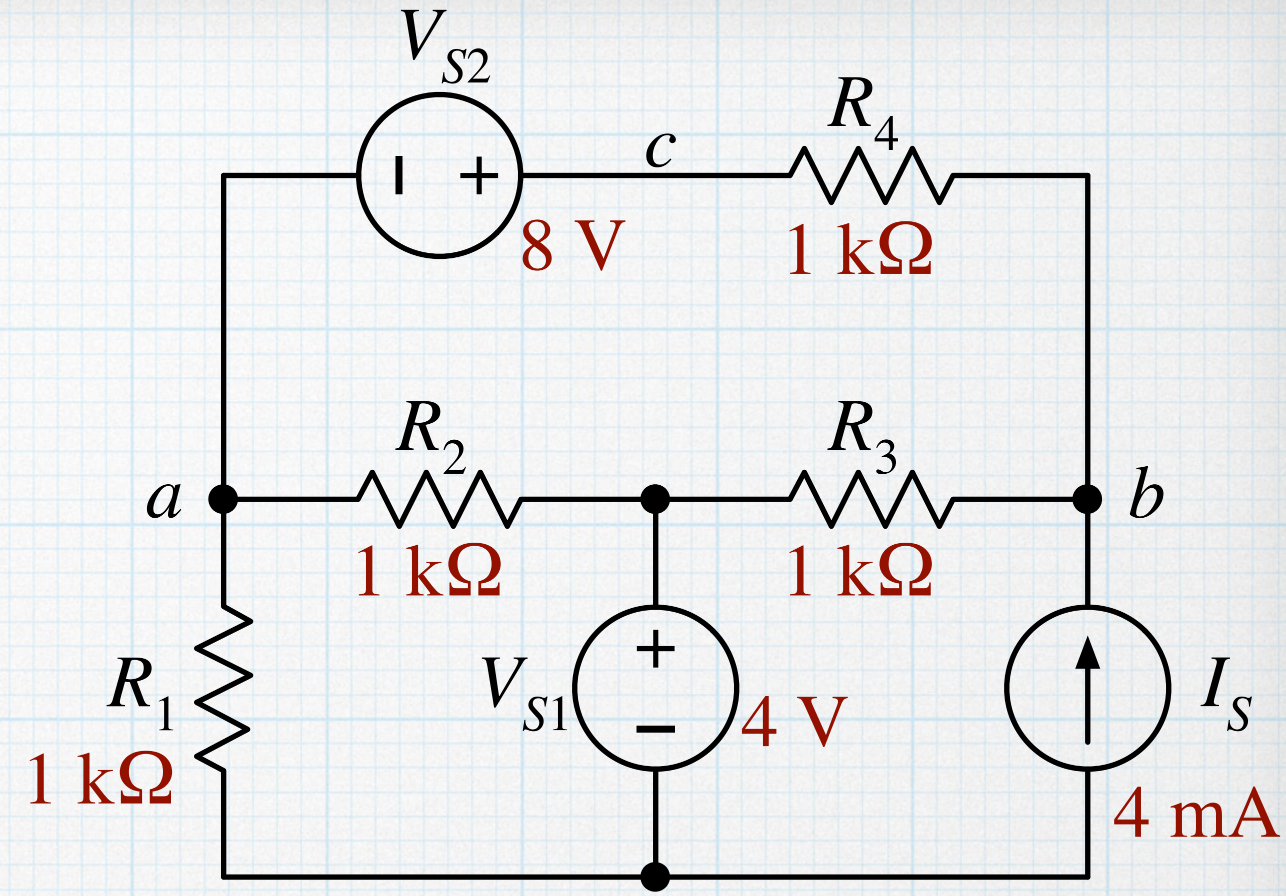
This gives us a single equation in two unknowns. Note that this is *exactly* the same situation that we encountered previously. To go any further, we once again need the auxiliary equation, $V_{S2} = v_b - v_a$. Now we have two equations in two unknowns, and the math proceeds as before.

There is nothing particularly "super" at the super node approach. It is a clever maneuver that allows us avoid one unknown in our system of equations. It is not a requirement to employ a super node.

Example 6

Use the node voltage method to find the current through R_4.

We need to find voltages on either side of R_4. The two voltage sources are not connected. This looks like another circuit that may need an auxiliary equation. (And we could use a super node.)

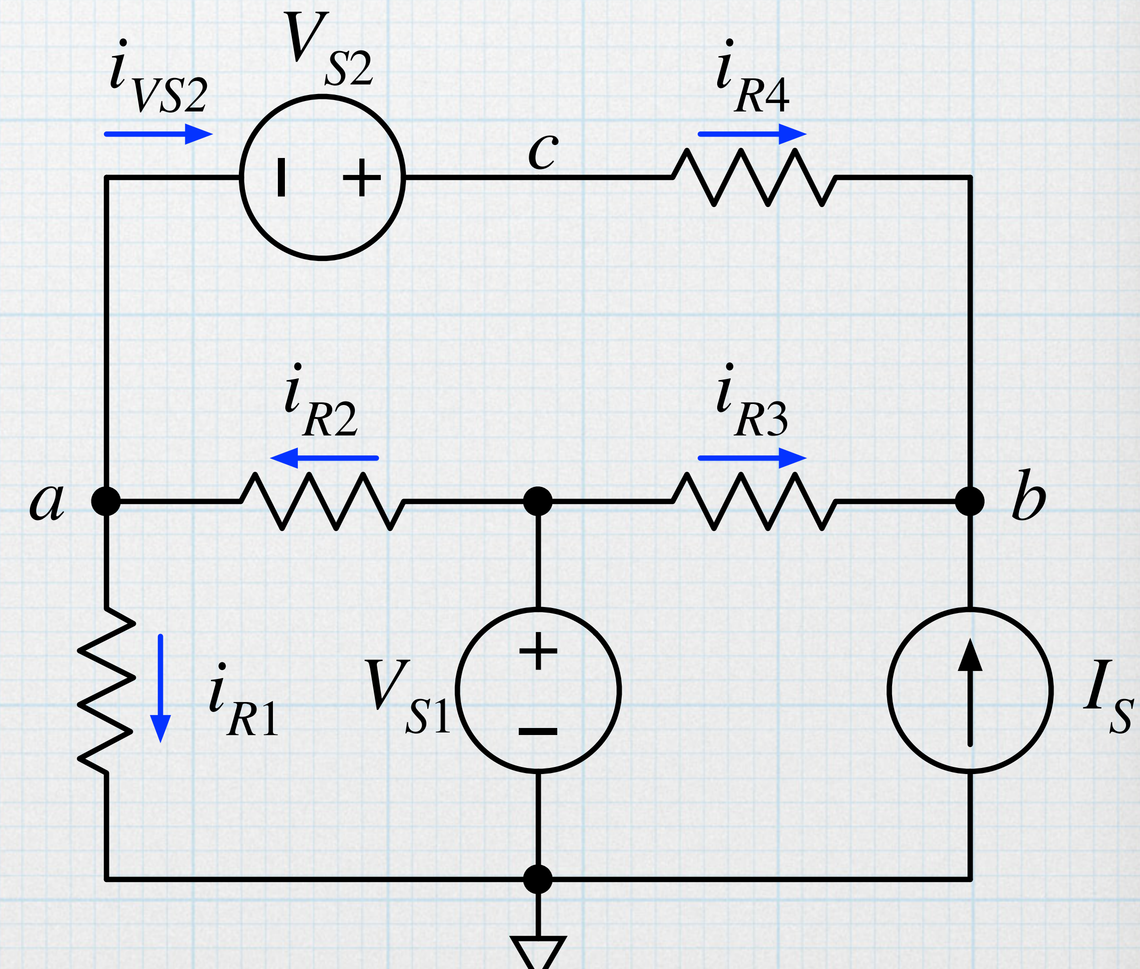


Choose the bottom node to be ground. Write KCL equations at the three other nodes.

$$a : i_{R2} = i_{R1} + i_{VS2}$$

$$b : i_{R3} + i_{R4} + I_S = 0$$

$$c : i_{VS2} = i_{R4}$$



Grrr. As expected i_{VS2} is causing trouble.

Example 6

Turn the KCL expressions into node-voltage equations.

$$a: \frac{V_{S1} - v_a}{R_2} = \frac{v_a}{R_1} + i_{VS2}$$

$$b: \frac{V_{S1} - v_b}{R_3} + \frac{v_c - v_b}{R_4} + I_S = 0$$

$$c: i_{VS2} = \frac{v_c - v_b}{R_4}$$

Three equations, four unknowns. We still need an auxiliary equation. Fortunately, one is readily at hand: $V_{S2} = v_c - v_a$. Now we have enough equations.

We could stuff the four equations into a solver and let it grind, but we can be more elegant that that. First, substitute the expression for i_{VS2} from c in to a.

$$a: \frac{V_{S1} - v_a}{R_2} = \frac{v_a}{R_1} + \frac{v_c - v_b}{R_4}$$

$$b: \frac{V_{S1} - v_b}{R_3} + \frac{v_c - v_b}{R_4} + I_S = 0$$

Example 6

Then we can substitute $v_c = V_{S2} + v_a$ at the appropriate spots, leaving us with two equations and two unknowns

$$a : \frac{V_{S1} - v_a}{R_2} = \frac{v_a}{R_1} + \frac{V_{S2} + v_a - v_b}{R_4}$$

$$b : \frac{V_{S1} - v_b}{R_3} + \frac{V_{S2} + v_a - v_b}{R_4} + I_S = 0$$

Re-arrange into a nicer form

$$\left(1 + \frac{R_2}{R_1} + \frac{R_2}{R_4}\right) v_a - \frac{R_2}{R_4} v_b = V_{S1} + \frac{R_2}{R_4} \cdot V_{S2}$$

$$-\frac{R_3}{R_4} \cdot v_a + \left(1 + \frac{R_3}{R_4}\right) v_b = V_{S1} + \frac{R_3}{R_4} \cdot V_{S2} + R_3 \cdot I_S$$

Insert numbers

$$3v_a - v_b = 12\,\mathrm{V}$$

$$-v_a + 2v_b = 16\,\mathrm{V}$$

Solve to give $v_a = 8$ V and $v_b = 12$ V.

Then $v_c = 16$ V and $i_{R4} = (16\text{ V} - 12\text{ V})\,/1\text{ k}\Omega = 4$ mA.

Example 6

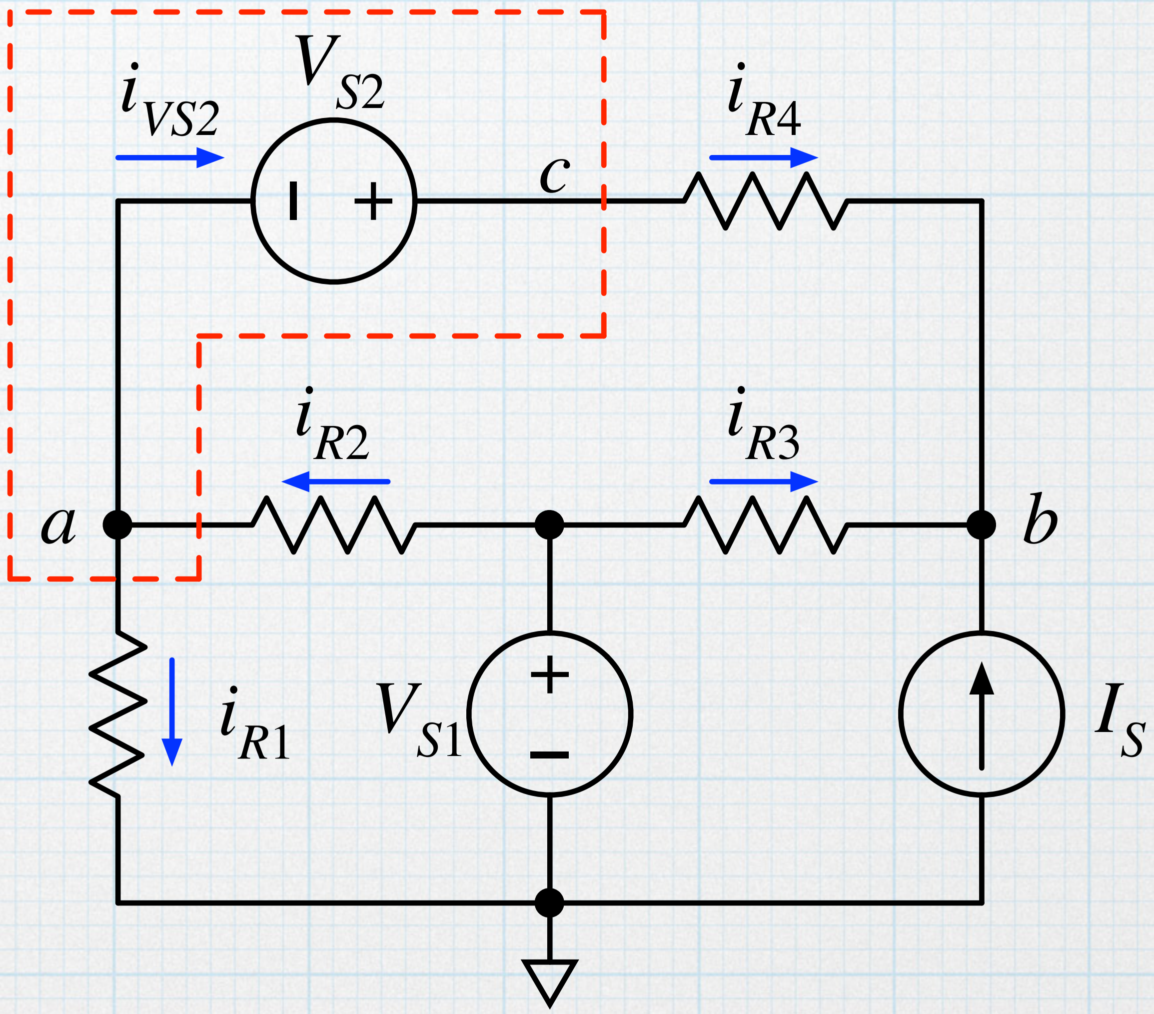
To use the super-node approach, we would draw a box around the second source and the nodes on either side. The box has a strange shape, but that's OK.

Balancing currents that are crossing the boundary of the box:

$$i_{R2} = i_{R1} + i_{R4}$$

No i_{VS2}! Turning this into a node-voltage equation:

$$\frac{V_{S1} - v_a}{R_2} = \frac{v_a}{R_1} + \frac{v_c - v_b}{R_4}$$



This is identical to equation *a* on the previous slide. We are on the same math path taken previously, and we would arrive at the same end result.